NOETHER'S THEOREM IN CLASSICAL MECHANICS

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 8 August 2021.

Coleman's Chapter 5 begins the study of symmetries and conservation laws, starting with Noether's theorem in classical mechanics. We suppose that the set of generalized coordinates q^a are transformed in a way characterized by a real parameter λ, so that

The generalized coordinates are assumed to be functions of time t.

$$q^a(t) \rightarrow q^a(t;\lambda) \tag{1}$$

The idea is that we transform the entire system in a fixed way, so we can envision such transformations as a fixed translation through space, a fixed rotation in space and so on.

If we consider an infinitesimal transformation characterized by an infinitesimal change $d\lambda$ in the parameter, then each coordinate will change according to a first order Taylor expansion:

$$q^a \rightarrow q^a + \left.\frac{\partial q^a}{\partial \lambda}\right|_{\lambda=0} d\lambda \tag{2}$$

The derivative is given a label:

$$Dq^a \equiv \left.\frac{\partial q^a}{\partial \lambda}\right|_{\lambda=0} \tag{3}$$

The change of the time derivatives of the coordinates is therefore

$$\dot{q}^a \rightarrow \dot{q}^a + \left.\frac{\partial \dot{q}^a}{\partial \lambda}\right|_{\lambda=0} d\lambda \tag{4}$$

$$= \dot{q}^a + D\dot{q}^a d\lambda \tag{5}$$

Since λ doesn't depend on time (for any given transformation, it's a constant) we have

$$D\dot{q}^a = \frac{d}{dt} Dq^a \tag{6}$$

We consider a Lagrangian $L(q^a, \dot{q}^a, t)$ that depends on the coordinates, their time derivatives and possibly explicitly on the time. The change under the above transformation is given by DL. However, I think what he meant to say is that the amount by which the Lagrangian changes is $DL \times d\lambda$, with

$$DL = \frac{\partial L}{\partial q^a} Dq^a + \frac{\partial L}{\partial \dot{q}^a} D\dot{q}^a \tag{7}$$

$$= \frac{\partial L}{\partial q^a} Dq^a + p_a D\dot{q}^a \tag{8}$$

where

$$p_a \equiv \frac{\partial L}{\partial \dot{q}^a} \tag{9}$$

is the momentum conjugate to the coordinate q^a.

The quantity DL, as written, doesn't depend on $d\lambda$, so I think we need to explicitly include this when we want the change in the Lagrangian. That is

$$L \to L + DL\, d\lambda \tag{10}$$

We now call a transformation a *symmetry* if and only if

$$DL = \frac{dF}{dt} \tag{11}$$

for some function $F(q^a, \dot{q}^a, t)$. If this is true, then the action between two times t_1 and t_2 is unchanged, so the equations of motion should also be unchanged. Remember that the equations of motion were obtained from the principle of least action, so if we change the Lagrangian in a way that doesn't change the action, the equations of motion should remain the same. The new action is

$$\mathcal{S}' = \int_{t_1}^{t_2} (L + DL\, d\lambda)\, dt \tag{12}$$

$$= \int_{t_1}^{t_2} L\, dt + d\lambda \int_{t_1}^{t_2} DL\, dt \tag{13}$$

$$= \mathcal{S} + d\lambda \int_{t_1}^{t_2} \frac{dF}{dt} dt \tag{14}$$

$$= \mathcal{S} + d\lambda \left(F(q_2^a, \dot{q}_2^a, t_2) - F(q_1^a, \dot{q}_1^a, t_1)\right) \tag{15}$$

where q_1^a is the coordinate q^a at time t_1. One of the conditions imposed when calculating the least action is that the generalized coordinates at the two extremes of time are fixed when the action is varied, so the the last term $\left(F(q_2^a, \dot{q}_2^a, t_2) - F(q_1^a, \dot{q}_1^a, t_1)\right)$ is zero when varied. That is

$$\delta F\left(q_2^a, \dot{q}_2^a, t_2\right) = 0 \quad (16)$$

$$\delta F\left(q_1^a, \dot{q}_1^a, t_1\right) = 0 \quad (17)$$

because the variations in q_1^a and q_2^a (and their time derivatives) are all zero at the endpoints of the path. In other words, the conditions $\delta\mathcal{S}' = 0$ and $\delta\mathcal{S} = 0$ give the same equations of motion.

We can now derive Noether's theorem by showing that if 11 is satisfied, then there is a quantity Q that is conserved (that is, Q remains constant over time). We define

$$Q = p_a Dq^a - F \quad (18)$$

and calculate its time derivative:

$$\frac{dQ}{dt} = \dot{p}_a Dq^a + p_a \frac{dDq^a}{dt} - \frac{dF}{dt} \quad (19)$$

We now use the Euler-Lagrange equation for classical particle mechanics:

$$\frac{\partial L}{\partial q^a} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) \quad (20)$$

Using 9, this is equivalent to

$$\dot{p}_a = \frac{\partial L}{\partial q^a} \quad (21)$$

so 19 becomes

$$\frac{dQ}{dt} = \frac{\partial L}{\partial q^a} Dq^a + p_a \frac{dDq^a}{dt} - \frac{dF}{dt} \quad (22)$$

From 8, we see that the first two terms on the RHS are DL so from that and 11 we have

$$\frac{dQ}{dt} = DL - DL = 0 \quad (23)$$

so that the quantity Q is conserved.

PINGBACKS

Pingback: Translation symmetry and conservation of momentum
Pingback: Time translation symmetry and conservation of energy
Pingback: Rotation symmetry and conservation of angular momentum
Pingback: Noether's theorem in classical field theory

Pingback: Currents from spacetime translations and the energy-momentum tensor