NORMAL ORDERING

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The real scalar field ϕ can be written as

$$\phi(x)=\int\frac{d^{3}\boldsymbol{p}}{\sqrt{(2\pi)^{3}}\sqrt{2\omega_{\boldsymbol{p}}}}\left[a_{\boldsymbol{p}}e^{-ip\cdot x}+a_{\boldsymbol{p}}^{\dagger}e^{ip\cdot x}\right] \tag{1}$$

We can use this to calculate the hamiltonian for a specific case, with Lagrangian

$$\mathscr{L}=\frac{1}{2}\left(\partial_{\mu}\phi\partial^{\mu}\phi-\mu^{2}\phi^{2}\right) \tag{2}$$

$$=\frac{1}{2}\left(\dot{\phi}^{2}-|\nabla\phi|^{2}-\mu^{2}\phi^{2}\right) \tag{3}$$

where the first minus sign is due to the relativistic metric.

The Hamiltonian is defined as

$$\mathscr{H}=\pi\dot{\phi}-\mathscr{L} \tag{4}$$

with the conjugate momentum defined as

$$\pi=\frac{\partial\mathscr{L}}{\partial(\partial_{0}\phi)} \tag{5}$$

$$=\partial_{0}\phi \tag{6}$$

$$=\dot{\phi} \tag{7}$$

The Hamiltonian thus becomes

$$\mathscr{H}=\frac{1}{2}\left(\dot{\phi}^{2}+|\nabla\phi|^{2}+\mu^{2}\phi^{2}\right) \tag{8}$$

The full Hamiltonian is the integral of $\mathscr{H}$ over all space, so we have

$$H=\int d^{3}\boldsymbol{x}\;\mathscr{H} \tag{9}$$

This is worked out using 1 by Coleman in eqns 4.56 through 4.62, with the result

$$H=\frac{1}{2}\int d^{3}\boldsymbol{p}\left(a_{\boldsymbol{p}}a_{\boldsymbol{p}}^{\dagger}+a_{\boldsymbol{p}}^{\dagger}a_{\boldsymbol{p}}\right)\omega_{\boldsymbol{p}} \tag{10}$$

Given that $a_{\boldsymbol{p}}^{\dagger}a_{\boldsymbol{p}}$ is the number operator N, we can write H in terms of N by using the commutator

$$\left[a_{\boldsymbol{p}},a_{\boldsymbol{p}'}^{\dagger}\right]=\delta\left(\boldsymbol{p}-\boldsymbol{p}'\right) \tag{11}$$

so we have

$$H=\int d^{3}\boldsymbol{p}\,\left(a_{\boldsymbol{p}}^{\dagger}a_{\boldsymbol{p}}+\frac{1}{2}\delta\left(0\right)\right)\omega_{\boldsymbol{p}} \tag{12}$$

The $\delta(0)$ term is infinite, so the result seems to indicate that we have infinite energy in the system. Various arguments are put forward to get around this problem. One approach is to say that in physics, all we care about are energy *differences*, so the infinite term, being a constant, cancels out whenever we calculate a difference.

The problem with this is that in some theories, such as general relativity, the absolute energy does matter, since it is the total energy density that determines the curvature of spacetime.

The ’solution’ that is usually adopted is to note that in moving from classical to quantum theory we are faced with an ambiguity in the ordering of operators. In classical theory, the order makes no difference since everything is described by ordinary numerical functions, and multiplication is commutative. In quantum theory, these functions become operators that don’t commute, so it does matter which order we place them in a product.

Although it looks like a fudge, the solution to the infinite energy is magicked away by imposing what is known as *normal ordering*. Normal ordering is defined only for free fields (I’m not sure why this restriction occurs; comments welcome). To apply it to a product of creation and annihilation operators, the prescription is that we place all creation operators to the left of all annihilation operators.

Normal ordering is usually indicated by enclosing a product of operators within a pair of colons. Thus we have

$$:a_{\boldsymbol{p}}a_{\boldsymbol{p}}^{\dagger}:=a_{\boldsymbol{p}}^{\dagger}a_{\boldsymbol{p}} \tag{13}$$

$$:a_{\boldsymbol{p}}^{\dagger}a_{\boldsymbol{p}}:=a_{\boldsymbol{p}}^{\dagger}a_{\boldsymbol{p}} \tag{14}$$

If we apply this to the Hamiltonian 10 we get

$$H =: \frac{1}{2} \int d^3\boldsymbol{p} \left(a_{\boldsymbol{p}} a_{\boldsymbol{p}}^{\dagger} + a_{\boldsymbol{p}}^{\dagger} a_{\boldsymbol{p}} \right) \omega_{\boldsymbol{p}} : \tag{15}$$

$$= \int d^3\boldsymbol{p}\, \omega_{\boldsymbol{p}} a_{\boldsymbol{p}}^{\dagger} a_{\boldsymbol{p}} \tag{16}$$

which is the more usual expression for the total energy. The number operator $a_{\boldsymbol{p}}^{\dagger} a_{\boldsymbol{p}}$ counts the number of particles with momentum $\boldsymbol{p}$ and integrand weights each of these numbers by the corresponding energy $\omega_{\boldsymbol{p}}$, with the integral summing everything up.

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