OUTER PRODUCT OF SPINORS

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The four solutions of the Dirac equation in relativistic quantum mechanics are

$$\left|\psi^{(1)}\right\rangle = \sqrt{\frac{E+m}{2m}}\left[\begin{array}{c} 1\\ 0\\ \frac{p^{3}}{E+m}\\ \frac{p^{1}+ip^{2}}{E+m} \end{array}\right]e^{-ipx}\equiv u_{1}e^{-ipx} \tag{1}$$

$$\left|\psi^{(2)}\right\rangle = \sqrt{\frac{E+m}{2m}}\left[\begin{array}{c} 0\\ 1\\ \frac{p^{1}-ip^{2}}{E+m}\\ -\frac{p^{3}}{E+m} \end{array}\right]e^{-ipx}\equiv u_{2}e^{-ipx} \tag{2}$$

$$\left|\psi^{(3)}\right\rangle = \sqrt{\frac{E+m}{2m}}\left[\begin{array}{c} \frac{p^{3}}{E+m}\\ \frac{p^{1}+ip^{2}}{E+m}\\ 1\\ 0 \end{array}\right]e^{ipx}\equiv v_{2}e^{ipx} \tag{3}$$

$$\left|\psi^{(4)}\right\rangle = \sqrt{\frac{E+m}{2m}}\left[\begin{array}{c} \frac{p^{1}-ip^{2}}{E+m}\\ -\frac{p^{3}}{E+m}\\ 0\\ 1 \end{array}\right]e^{ipx}\equiv v_{1}e^{ipx} \tag{4}$$

We can derive a useful relation by calculating the outer product

$$\sum_{r}u_{r}\overline{u}_{r} \tag{5}$$

where

$$\overline{u}_{r}\equiv u_{r}^{\dagger}\gamma^{0} \tag{6}$$

Here, the gamma matrices are

$$\gamma^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{7}$$

$$\gamma^1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \tag{8}$$

$$\gamma^2 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} \tag{9}$$

$$\gamma^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{10}$$

To work out 5, we have

$$u_1^\dagger = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} 1 & 0 & \frac{p^3}{E+m} & \frac{p^1-ip^2}{E+m} \end{bmatrix} \tag{11}$$

$$u_2^\dagger = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} 0 & 1 & \frac{p^1+ip^2}{E+m} & -\frac{p^3}{E+m} \end{bmatrix} \tag{12}$$

We have

$$u_1\overline{u}_1 = u_1 u_1^\dagger \gamma^0 \tag{13}$$

$$= \frac{E+m}{2m}\begin{bmatrix} 1 \\ 0 \\ \frac{p^3}{E+m} \\ \frac{p^1+ip^2}{E+m} \end{bmatrix}\begin{bmatrix} 1 & 0 & \frac{p^3}{E+m} & \frac{p^1-ip^2}{E+m} \end{bmatrix}\gamma^0 \tag{14}$$

$$\frac{E+m}{2m}\begin{bmatrix} 1 & 0 & \frac{p^3}{E+m} & \frac{p^1-ip^2}{E+m} \\ 0 & 0 & 0 & 0 \\ \frac{p^3}{E+m} & 0 & \left(\frac{p^3}{E+m}\right)^2 & \frac{p^3\left(p^1-ip^2\right)}{(E+m)^2} \\ \frac{p^1+ip^2}{E+m} & 0 & \frac{p^3\left(p^1+ip^2\right)}{(E+m)^2} & \frac{\left(p^1\right)^2+\left(p^2\right)^2}{(E+m)^2} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{15}$$

$$= \frac{E+m}{2m}\begin{bmatrix} 1 & 0 & -\frac{p^3}{E+m} & \frac{-p^1+ip^2}{E+m} \\ 0 & 0 & 0 & 0 \\ \frac{p^3}{E+m} & 0 & -\left(\frac{p^3}{E+m}\right)^2 & \frac{p^3\left(-p^1+ip^2\right)}{(E+m)^2} \\ \frac{p^1+ip^2}{E+m} & 0 & -\frac{p^3\left(p^1+ip^2\right)}{(E+m)^2} & \frac{-\left(p^1\right)^2-\left(p^2\right)^2}{(E+m)^2} \end{bmatrix} \tag{16}$$

Similarly, we have

$$u_2\overline{u}_2 = u_2 u_2^\dagger \gamma^0 \tag{17}$$

$$= \frac{E+m}{2m}\begin{bmatrix} 0 \\ 1 \\ \frac{p^1-ip^2}{E+m} \\ -\frac{p^3}{E+m}\end{bmatrix}\begin{bmatrix} 0 & 1 & \frac{p^1+ip^2}{E+m} & -\frac{p^3}{E+m}\end{bmatrix}\gamma^0 \tag{18}$$

$$\frac{E+m}{2m}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{p^1+ip^2}{E+m} & -\frac{p^3}{E+m} \\ 0 & \frac{p^1-ip^2}{E+m} & \frac{\left(p^1\right)^2+\left(p^2\right)^2}{(E+m)^2} & -\frac{p^3\left(p^1-ip^2\right)}{(E+m)^2} \\ 0 & -\frac{p^3}{E+m} & -\frac{p^3\left(p^1+ip^2\right)}{(E+m)^2} & \left(\frac{p^3}{E+m}\right)^2\end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1\end{bmatrix} \tag{19}$$

$$= \frac{E+m}{2m}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{-p^1-ip^2}{E+m} & \frac{p^3}{E+m} \\ 0 & \frac{p^1-ip^2}{E+m} & \frac{-\left(p^1\right)^2-\left(p^2\right)^2}{(E+m)^2} & \frac{p^3\left(p^1-ip^2\right)}{(E+m)^2} \\ 0 & -\frac{p^3}{E+m} & \frac{p^3\left(p^1+ip^2\right)}{(E+m)^2} & -\left(\frac{p^3}{E+m}\right)^2\end{bmatrix} \tag{20}$$

Adding 16 and 20 we get

$$\sum_r u_r\overline{u}_r = \frac{E+m}{2m}\begin{bmatrix} 1 & 0 & -\frac{p^3}{E+m} & \frac{-p^1+ip^2}{E+m} \\ 0 & 1 & \frac{-p^1-ip^2}{E+m} & \frac{p^3}{E+m} \\ \frac{p^3}{E+m} & \frac{p^1-ip^2}{E+m} & \frac{-\left(p^1\right)^2-\left(p^2\right)^2-\left(p^3\right)^2}{(E+m)^2} & 0 \\ \frac{p^1+ip^2}{E+m} & -\frac{p^3}{E+m} & 0 & \frac{-\left(p^1\right)^2-\left(p^2\right)^2-\left(p^3\right)^2}{(E+m)^2}\end{bmatrix} \tag{21}$$

$$= \frac{E+m}{2m}\begin{bmatrix} 1 & 0 & -\frac{p^3}{E+m} & \frac{-p^1+ip^2}{E+m} \\ 0 & 1 & \frac{-p^1-ip^2}{E+m} & \frac{p^3}{E+m} \\ \frac{p^3}{E+m} & \frac{p^1-ip^2}{E+m} & \frac{-\boldsymbol{p}^2}{(E+m)^2} & 0 \\ \frac{p^1+ip^2}{E+m} & -\frac{p^3}{E+m} & 0 & \frac{-\boldsymbol{p}^2}{(E+m)^2}\end{bmatrix} \tag{22}$$

where $\boldsymbol{p}$ is the 3-momentum.

We can find a simpler expression for this if we consider

$$\frac{\not{p}+m}{2m} = \frac{1}{2m}\left(p_\mu\gamma^\mu + m\right) \tag{23}$$

First, we recall that lowering a spatial index introduces a minus sign, so that

$$p_i = -p^i \tag{24}$$

We can then plug in the gamma matrices from above to get

$$\frac{\not{p}+m}{2m} = \frac{1}{2m}\left(p_0\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix} + p_1\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{bmatrix} + p_2\begin{bmatrix}0&0&0&-i\\0&0&i&0\\0&i&0&0\\-i&0&0&0\end{bmatrix} + p_3\begin{bmatrix}0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0\end{bmatrix} + m\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\right) \tag{25}$$

where we've multiplied the final m by the identity matrix because this is a matrix equation.

Since $p_0 = E$, we have

$$\frac{\not{p}+m}{2m} = \frac{1}{2m}\begin{bmatrix}E+m&0&p_3&p_1-ip_2\\0&E+m&p_1+ip_2&-p_3\\-p_3&-p_1+ip_2&-E+m&0\\-p_1-ip_2&p_3&0&-E+m\end{bmatrix} \tag{26}$$

$$= \frac{1}{2m}\begin{bmatrix}E+m&0&-p^3&-p^1+ip^2\\0&E+m&-p^1-ip^2&p^3\\p^3&p^1-ip^2&-E+m&0\\p^1+ip^2&-p^3&0&-E+m\end{bmatrix} \tag{27}$$

where in the last line we raised the index on the p_i components.

We can now multiply all the components except for the upper-left 2×2 block by $\frac{E+m}{E+m}$, and use

$$(E+m)(-E+m) = -\left(E^2-m^2\right) = -\boldsymbol{p}^2 \tag{28}$$

We then get

$$\frac{\not{p}+m}{2m} = \frac{1}{2m}\begin{bmatrix}E+m&0&-p^3\frac{E+m}{E+m}&\left(-p^1+ip^2\right)\frac{E+m}{E+m}\\0&E+m&\left(-p^1-ip^2\right)\frac{E+m}{E+m}&p^3\frac{E+m}{E+m}\\p^3\frac{E+m}{E+m}&\left(p^1-ip^2\right)\frac{E+m}{E+m}&\frac{-\boldsymbol{p}^2}{E+m}&0\\\left(p^1+ip^2\right)\frac{E+m}{E+m}&-p^3\frac{E+m}{E+m}&0&\frac{-\boldsymbol{p}^2}{E+m}\end{bmatrix} \tag{29}$$

If we now pull out a factor of $E+m$ from all matrix elements, we have

$$\frac{\not{p}+m}{2m}=\frac{E+m}{2m}\left[\begin{array}{cccc} 1 & 0 & -\frac{p^{3}}{E+m} & \frac{-p^{1}+ip^{2}}{E+m}\\ 0 & 1 & \frac{-p^{1}-ip^{2}}{E+m} & \frac{p^{3}}{E+m}\\ \frac{p^{3}}{E+m} & \frac{p^{1}-ip^{2}}{E+m} & \frac{-\boldsymbol{p}^{2}}{(E+m)^{2}} & 0\\ \frac{p^{1}+ip^{2}}{E+m} & -\frac{p^{3}}{E+m} & 0 & \frac{-\boldsymbol{p}^{2}}{(E+m)^{2}} \end{array}\right] \tag{30}$$

which matches 22. Thus we have the relation

$$\boxed{\sum_{r}u_{r}\overline{u}_{r}=\frac{\not{p}+m}{2m}} \tag{31}$$