PHASE SPACE AND THE S MATRIX

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 19 November 2021.

In section 11.4, Coleman begins the process of relating the scattering amplitude derived from Feynman diagrams to the realm of scattering cross-sections that are measured by experimentalists. To do this, he considers a system initially constrained to be bounded by a cube of side length L and volume $V = L^3$. This is essentially a 3-d infinite square well. In rectangular coordinates, the allowable 3-d momenta are then

$$p_i = \frac{2\pi}{L} n_i \tag{1}$$

where $i = 1, 2, 3$ for the three dimensions.

From here, Coleman uses analogies with his derivations in infinite space. So it's a matter of comparing the results for particles in a box with those of a free particle in infinite space, and justifying the boxed versions.

Coleman introduced boxed states in §2.2 on occupation number representation. The normalization of boxed momentum states is

$$\left\langle \boldsymbol{p} \middle| \boldsymbol{p}' \right\rangle = \delta_{\boldsymbol{p}\boldsymbol{p}'} \tag{2}$$

with commutators for the creation and annihilation operators given by

$$\left[a_{\boldsymbol{p}}, a_{\boldsymbol{p}'}^{\dagger}\right] = \delta_{\boldsymbol{p}\boldsymbol{p}'} \tag{3}$$

with all other commutators equal to zero.

In free space, the scalar quantum field is given by

$$\phi(x) = \int \frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3}\sqrt{2\omega_{\boldsymbol{p}}}} \left[a_{\boldsymbol{p}} e^{-ip\cdot x} + a_{\boldsymbol{p}}^{\dagger} e^{ip\cdot x}\right] \tag{4}$$

This gives a commutator as shown in Coleman's equation 3.61:

$$\left[\dot{\phi}(\boldsymbol{x}, t), \phi(\boldsymbol{y}, t)\right] = -i\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}) \tag{5}$$

To get the same commutator from boxed states, we can use as the quantum field Coleman's equation 11.41:

$$\phi(x)=\sum_{\boldsymbol{p}}\frac{1}{\sqrt{V}}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\left[a_{\boldsymbol{p}}e^{-ip\cdot x}+a_{\boldsymbol{p}}^{\dagger}e^{ip\cdot x}\right] \tag{6}$$

With this form, we can use the expression for the one dimensional delta function given in footnote 10:

$$\delta(x)=\frac{1}{L}\sum_{n=-\infty}^{\infty}e^{-i(2\pi n/L)x} \tag{7}$$

$$=\frac{1}{L}\sum_{n=-\infty}^{\infty}e^{-ipx} \tag{8}$$

This generalizes to 3-d to give

$$\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})=\frac{1}{V}\sum_{\boldsymbol{p}}e^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} \tag{9}$$

From 6 we have, since $p^0=E_{\boldsymbol{p}}$:

$$\dot{\phi}(x)=i\sum_{\boldsymbol{p}}\frac{1}{\sqrt{V}}\sqrt{\frac{E_{\boldsymbol{p}}}{2}}\left[-a_{\boldsymbol{p}}e^{-ip\cdot x}+a_{\boldsymbol{p}}^{\dagger}e^{ip\cdot x}\right] \tag{10}$$

We have, at equal times (so that $x^0=y^0=t$):

$$\left[\dot{\phi}(\boldsymbol{x},t),\phi(\boldsymbol{y},t)\right]=i\sum_{\boldsymbol{p}}\frac{1}{\sqrt{V}}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\sum_{\boldsymbol{p}'}\frac{1}{\sqrt{V}}\sqrt{\frac{E_{\boldsymbol{p}'}}{2}}$$

$$\left[-\delta_{\boldsymbol{p}\boldsymbol{p}'}\left(e^{-ip\cdot x+ip'\cdot y}+e^{ip\cdot x-ip'\cdot y}\right)\right]_{x^0=y^0} \tag{11}$$

$$=\frac{-i}{2V}\sum_{\boldsymbol{p}}\left(e^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}+e^{i\boldsymbol{p}\cdot(\boldsymbol{y}-\boldsymbol{x})}\right) \tag{12}$$

Since the delta function is even, the two terms in the sum each contribute the same delta function to the result, so we have

$$\left[\dot{\phi}(\boldsymbol{x},t),\phi(\boldsymbol{y},t)\right]=-i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \tag{13}$$

which agrees with 5.

The next stage of the argument discusses the normalization of the states. Coleman considers only states with either one or two particles. For one particle, the initial state is just

$$|i\rangle=|\boldsymbol{p}\rangle \tag{14}$$

For two particles, he argues that the correct state is

$$\left|i\right\rangle = \sqrt{V}\left|\boldsymbol{p}_1,\boldsymbol{p}_2\right\rangle \tag{15}$$

The argument isn't entirely clear to me, but basically he is saying that the probability that each particle is somewhere in the box is 1 (since we're restricting the universe to our cubical box, so it contains everything we're considering, namely the two particles). As we increase the size of the box, he argues that if we didn't have the factor of $\sqrt{V}$, it's less and less likely that the two particles would ever be close enough to interact, so nothing would happen. By putting in a factor $\sqrt{V}$ and remembering that it's the square of a state that turns up in measurements, we multiply the bare probability by the volume V, so as the volume increases the probability per unit volume of finding a particle remains constant.

Next, review the calculation we did earlier to evaluate the Feynman diagrams. This involved calculating a number of terms of the form

$$\left\langle 0\left|F\right|p_1,p_2\right\rangle \tag{16}$$

where F is some term arising from the field operators.

In our boxed model, we need to consider the quantity $\left\langle 0\left|\psi(x)\right|\boldsymbol{p}\right\rangle$. We recall that a relativistic ket is given in terms of the 3-d momentum ket by a factor resulting from the correct relativistic integration measure:

$$\left|p\right\rangle = \sqrt{(2\pi)^3}\sqrt{2\omega_{\boldsymbol{p}}}\left|\boldsymbol{p}\right\rangle \tag{17}$$

From 6 we have

$$\left\langle 0\left|\psi(x)\right|\boldsymbol{p}\right\rangle = \left\langle 0\right|\sum_{\boldsymbol{p}'}\frac{1}{\sqrt{V}}\frac{1}{\sqrt{2E_{\boldsymbol{p}'}}}\left[a_{\boldsymbol{p}'}e^{-ip'\cdot x}+a_{\boldsymbol{p}'}^{\dagger}e^{ip'\cdot x}\right]\left|\boldsymbol{p}\right\rangle \tag{18}$$

Since we're going from a one-particle state to the vacuum, only the annihilation operator in this sum will give a non-zero contribution, so we have

$$\left\langle 0\left|\psi(x)\right|\boldsymbol{p}\right\rangle = \frac{1}{\sqrt{V}}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}e^{-ip\cdot x} \tag{19}$$

In Coleman's original discussion of the evaluation of Feynman diagrams, he arrived at the formula

$$\left\langle p_1'p_2'\left|W_2\right|p_1p_2\right\rangle = (2\pi)^4\delta^{(4)}\left(p_f-p_i\right)i\mathcal{A}_{fi} \tag{20}$$

where W_2 is the 2nd order term in the S matrix. By analogy with this, he writes the transition for the boxed model as equation 11.46:

$$\langle f|S-1|i\rangle = i\mathcal{A}_{fi}^{VT}(2\pi)^4\delta_{VT}^{(4)}(p_f-p_i)\prod_{\text{final}}\left[\frac{1}{\sqrt{2VE_{\boldsymbol{p}_f}}}\right]\prod\left[\frac{1}{\sqrt{2E_{\boldsymbol{p}_i}}}\right]\frac{1}{\sqrt{V}} \tag{21}$$

The notation VT means that these quantities aren't the same as those that appear in 20. However, in the limit of $V,T\to\infty$, they should reduce to these terms. In particular, the amplitude should go as

$$\lim_{V,T\to\infty}\mathcal{A}_{fi}^{VT} = \mathcal{A}_{fi} \tag{22}$$

The extra product terms arise from the fact that we're using $|\boldsymbol{p}\rangle$ states rather than the $|p\rangle$ states that we did when evaluating Feynman diagrams. The factors of $1/\sqrt{2VE_{\boldsymbol{p}}}$ come from 19. There is only one $1/\sqrt{V}$ for the initial particles in 21 because of the normalization 15. We're assuming that there are two incoming particles, so one of the $1/\sqrt{V}$ factors is cancelled by the normalization of the initial two particle state. The expression 21 places no restriction on the number of outgoing particles in the final state.

The function $\delta_{VT}^{(4)}(p_f-p_i)$ isn't strictly speaking a delta function, since it is its square that turns up in scattering cross sections. Coleman shows in equations 11.49 through 11.52 that this function satisfies

$$\lim_{V,T\to\infty}\left[(2\pi)^4\delta_{VT}^{(4)}(p)\right]^2 = (2\pi)^4 VT\delta^{(4)}(p) \tag{23}$$

Next, we must consider how many states there are in an infinitesimal volume $d^3\boldsymbol{p}$ of momentum space. In our boxed model, the momentum states are spaced a distance of $\frac{2\pi}{L}$ apart in each of the three dimensions. In a unit volume (a cube with side length 1), there are

$$\left(\frac{2\pi}{L}\right)^{-3} = \left(\frac{L}{2\pi}\right)^3 \tag{24}$$

allowable momentum points. Thus as the size L of the box increases, the momentum points become closer to each other. A volume $d^3\boldsymbol{p}$ will therefore contain

$$\left(\frac{L}{2\pi}\right)^3 d^3\boldsymbol{p} = \frac{V}{(2\pi)^3}d^3\boldsymbol{p} \tag{25}$$

In a real scattering experiment, we are interested only in the transition amplitude into a particular narrow region of momentum space. This is known as the *differential transition probability* and is the transition probability calculated from the matrix element 21 multiplied by the number of momentum points 25 in our desired volume.

Coleman writes this out in equations 11.54 to 11.58. When all the terms are multiplied together, the volume V cancels out, so we can take the limit $V \to \infty$ safely. There is still a factor of T for the total time over which the scattering occurs, but the model above assumes that we have an interaction that is always on, and that our incoming particles are described by plane waves (so they are everywhere in space). We can divide out T to get the probability per unit time. The final result is Coleman's equations 11.57 and 11.58:

$$\boxed{\frac{\text{differential transition prob}}{\text{unit time}} = \left|\mathcal{A}_{fi}\right|^2 D \prod_{\text{initial}} \frac{1}{2E_i}} \tag{26}$$

where D is called the *relativistic density of final states* and is given by

$$\boxed{D \equiv (2\pi)^4 \delta^{(4)}\left(p_f - p_i\right) \prod_{\text{final}} \frac{d^3\boldsymbol{p}_f}{(2\pi)^3 2E_f}} \tag{27}$$

One important feature of this result is that we have one factor of 2π in the numerator for each delta function ($\delta^{(4)}$ is a 4-d delta function so counts as 4 delta functions, hence has a factor of $(2\pi)^4$ attached to it), and one factor of 2π in the denominator for each dp. Note that the product in 27 contains 3 dps for *each* particle in the final state, so there is a factor of $1/(2\pi)^3$ for each particle in the final state.

PINGBACKS

Pingback: Decay of a single particle