# POLARIZATION VECTORS

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The 4d wave equation (Maxwell's equations in Lorenz gauge) has the general solution

$$A^{\mu}(x)=\sum_{r,\boldsymbol{k}}\frac{1}{\sqrt{2V\omega_{\boldsymbol{k}}}}\varepsilon_{r}^{\mu}A_{r}(\boldsymbol{k})e^{-ikx}+\sum_{r,\boldsymbol{k}}\frac{1}{\sqrt{2V\omega_{\boldsymbol{k}}}}\varepsilon_{r}^{\mu}A_{r}^{\dagger}(\boldsymbol{k})e^{ikx} \tag{1}$$

Here, $A_r(\boldsymbol{k})$ and $B_r^{\dagger}(\boldsymbol{k})$ are (possibly complex) numerical coefficients. The $\varepsilon_r^{\mu}$ are the polarization vectors, which serve as a basis for the 4d space. We can choose any 4 independent vectors to serve as $\varepsilon_r$.

One choice is to align $\varepsilon_3$ with $\boldsymbol{k}$, the direction of travel of the photon. Although there is a sum over $\boldsymbol{k}$ in 1, we can consider situations where the field is travelling in a single direction, so all the $\boldsymbol{k}$ vectors are parallel. The vectors can have different frequencies, which we would derive from the boundary conditions on the volume $V$ in which the field is contained.

With this choice, we have

$$\varepsilon_3=\frac{\boldsymbol{k}}{|\boldsymbol{k}|} \tag{2}$$

If we choose the other polarization vectors to be orthogonal, then

$$\varepsilon_1\cdot\boldsymbol{k}=\varepsilon_2\cdot\boldsymbol{k}=0 \tag{3}$$

Since electromagnetic waves are transverse waves, the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are perpendicular to $\boldsymbol{k}$, so the vectors $\varepsilon_1$ and $\varepsilon_2$ can be chosen to be parallel to $\boldsymbol{E}$ and $\boldsymbol{B}$.

To see how this works, suppose we also align the coordinate system so that $\varepsilon_i$ points along the $x^i$ direction. Now consider the case where $A^{\mu}$ is polarized along the $x^1$ direction, and consider the spatial part of 1. Also taking $A_1$ to be real, so that $A_1^{\dagger}=A_1$, we have, considering only a single vector $\boldsymbol{k}$:

$$
\begin{aligned}
A^{i}(x) &= \frac{1}{\sqrt{2V\omega_{\boldsymbol{k}}}}\varepsilon_{1}^{i}A_{1}(\boldsymbol{k})\left[e^{-ikx}+e^{ikx}\right] &(4)\\
&= \sqrt{\frac{2}{V\omega_{\boldsymbol{k}}}}\varepsilon_{1}^{i}A_{1}(\boldsymbol{k})\cos kx &(5)
\end{aligned}
$$

Since the wave is propagating in the $x^3 = z$ direction, we have

$$
kx = \omega_{\boldsymbol{k}}t - pz \tag{6}
$$

where $p = |\boldsymbol{p}|$ is the magnitude of the momentum. Also

$$
\varepsilon_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \tag{7}
$$

We can now find the magnetic field $\boldsymbol{B}$ from

$$
\begin{aligned}
\boldsymbol{B} &= \nabla\times\boldsymbol{A} &(8)\\
&= \sqrt{\frac{2}{V\omega_{\boldsymbol{k}}}}A_{1}(\boldsymbol{k})\begin{vmatrix} \hat{x} & \hat{y} & \hat{z}\\ \partial_x & \partial_y & \partial_z\\ \cos kx & 0 & 0 \end{vmatrix} &(9)\\
&= \sqrt{\frac{2}{V\omega_{\boldsymbol{k}}}}A_{1}(\boldsymbol{k})\left[\hat{y}\left(\partial_z\cos kx\right)+\hat{z}\left(\partial_y\cos kx\right)\right] &(10)\\
&= \sqrt{\frac{2}{V\omega_{\boldsymbol{k}}}}A_{1}(\boldsymbol{k})\,\hat{y}\left(p\sin kx\right)+0 &(11)\\
&= \sqrt{\frac{2}{V\omega_{\boldsymbol{k}}}}A_{1}(\boldsymbol{k})\,p\sin kx\;\hat{y} &(12)
\end{aligned}
$$

where the $+0$ in the third line is because $kx$ does not depend on $y$ (see 6). Thus the magnetic field is parallel to $\hat{y}$ which is also the $x^2$ direction, and is perpendicular to $\boldsymbol{A}$.

The electric field can be found from

$$\nabla\times\boldsymbol{E}=\nabla\times\left(-\nabla\Phi-\frac{\partial\boldsymbol{A}}{\partial t}\right) \tag{13}$$

$$=-\frac{\partial}{\partial t}\left(\nabla\times\boldsymbol{A}\right) \tag{14}$$

$$=-\frac{\partial}{\partial t}\boldsymbol{B} \tag{15}$$

$$=-\sqrt{\frac{2\omega_{\boldsymbol{k}}}{V}}A_1\left(\boldsymbol{k}\right)p\cos kx\;\hat{y} \tag{16}$$

since curl of grad is always zero. Thus the curl of $\boldsymbol{E}$ is parallel to the curl of $\boldsymbol{A}$ which is true if $\boldsymbol{E}$ is parallel to $\boldsymbol{A}$, so it lies in the $x^1$ direction. Both $\boldsymbol{E}$ and $\boldsymbol{B}$ are perpendicular to $\boldsymbol{k}$, so the wave is transverse, as required.

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