POSITION AND VELOCITY AS INDEPENDENT VARIABLES

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The principle of least action gives the equations of motion of a system in terms of the Lagrangian. In the derivation, the generalized coordinates are q_i and the generalized velocities are $\dot{q}_i$. These are taken as independent variables in the derivation of least action.

$$\boxed{\frac{\partial L}{\partial q_i}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)=0} \tag{1}$$

Why can we take q_i and $\dot{q}_i$ as independent variables? It would seem that they are *not* independent since once you've specified q_i you can get $\dot{q}_i$ by just taking the derivative. [As an aside, I've never seen this question addressed in any of the textbooks I've looked at. They all just assume that we can take q_i and $\dot{q}_i$ as independent variables without explaining why.]

It's perhaps easiest to understand the idea of independent variables by considering a simple case of an ordinary function in the 2-d $x-y$ plane, such as $y=x^2$. With this function, once we specify x, y is determined, so in this case, y is certainly not independent of x. However, suppose we define another function such as

$$L=f(x,y)=y-x^2 \tag{2}$$

In this case, both x and y are independent, as either of them can take on any value we choose. We can obtain different curves (in this case, they are all parabolas) by requiring $f(x,y)$ to be equal to various constants. To obtain the specific curve $y=x^2$, we must impose a specific constraint on L, namely that $f(x,y)=0$.

The same logic applies to the Lagrangian, except that the constraint is a bit more elaborate. If we consider a single particle at a given instant, its position is given by q_i (which can consist of up to 3 components, depending on how many dimensions we're considering), and its velocity by v_i (which again can have up to 3 components). If all we're doing is specifying the position and velocity of a particle at one point, then q_i and v_i can take on any values so, at this stage, they *are* technically independent variables. This is the starting point of the Lagrangian. The Lagrangian is given as a function of the positions q_i and velocities v_i, with no constraints applied on how these two variables relate to each other. At this stage, we don't know the path $q_i(t)$ followed by the particle, so we also don't know its velocity.

The constraint is to apply the principle of least action. This leads to the Euler-Lagrange equations 1, the solution of which gives the path $q_i(t)$ followed by the particle. After this constraint has been applied, the velocity is now given by $v_i(t) = \dot{q}_i(t)$, and now q_i and $\dot{q}_i$ are no longer independent, as $\dot{q}_i(t)$ is just the derivative of q_i. What can cause confusion is that the Lagrangian is usually introduced as $L(q_i, \dot{q}_i)$ rather than, say, $L(q_i, v_i)$, so it's easy to interpret the variable $\dot{q}_i$ as the derivative of q_i *before* the constraint of least action is applied, and this isn't true. The connection of q_i and $\dot{q}_i$ emerges only after we solve the Euler-Lagrange equations.

Since the equations of motion 1 in general are second-order differential equations, their solution requires two initial conditions, which are usually the values of the position and velocity at some specific time, such as $t = 0$.

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