PROBABILITY OF PARTICLE OUTSIDE LIGHT CONE

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In Coleman's lectures, section 1.3, he does a calculation to see if there is a non-zero probability of finding a particle outside its light cone if we use the hamiltonian based on special relativity, which is

$$\omega_{\boldsymbol{p}}=\sqrt{p^2+\mu^2} \tag{1}$$

We start with a particle localized at the origin at time $t=0$, which is given by

$$\langle\boldsymbol{x}|\psi\rangle=\delta(\boldsymbol{x}) \tag{2}$$

This is equivalent to saying that the particle's state $|\psi\rangle$ is an eigenstate of position. The momentum space version of the position eigenstate is

$$\langle\boldsymbol{p}|\boldsymbol{x}\rangle=\frac{1}{(2\pi)^{3/2}}e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} \tag{3}$$

The momentum space wave function can be found by inserting a complete set of position states, so we have

$$\begin{aligned}
\langle\boldsymbol{p}|\psi\rangle &= \int d^3x\ \langle\boldsymbol{p}|\boldsymbol{x}\rangle\langle\boldsymbol{x}|\psi\rangle && (4)\\
&= \frac{1}{(2\pi)^{3/2}}\int d^3x\ e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\delta(\boldsymbol{x}) && (5)\\
&= \frac{1}{(2\pi)^{3/2}}e^{-i\boldsymbol{p}\cdot\boldsymbol{0}} && (6)\\
&= \frac{1}{(2\pi)^{3/2}} && (7)
\end{aligned}$$

The momentum space function is a constant, which means it has no dependence on $\boldsymbol{p}$, so the particle could have any momentum with equal probability.

To find the behaviour of the particle at future times, we can use the propagator relation:

$$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle \tag{8}$$

Thus the probability amplitude for finding the particle at position $\boldsymbol{x}$ at time t is given by

$$P(\boldsymbol{x},t) = \left\langle \boldsymbol{x} \left| e^{-iHt} \right| \psi \right\rangle \tag{9}$$

Inserting a complete set of momentum states, we can expand this as

$$\left\langle \boldsymbol{x} \left| e^{-iHt} \right| \psi \right\rangle = \int d^3p \, \left\langle \boldsymbol{x} \left| e^{-iHt} \right| \boldsymbol{p} \right\rangle \langle \boldsymbol{p} | \psi \rangle \tag{10}$$

The momentum state $|\boldsymbol{p}\rangle$ is an eigenstate of the hamiltonian H with the eigenvalue equal to the energy of that state, which is $\omega_{\boldsymbol{p}}$. We therefore have

$$\left\langle \boldsymbol{x} \left| e^{-iHt} \right| \psi \right\rangle = \int d^3p \, e^{-i\omega_{\boldsymbol{p}}t} \langle \boldsymbol{x} | \boldsymbol{p} \rangle \langle \boldsymbol{p} | \psi \rangle \tag{11}$$

$$= \frac{1}{(2\pi)^3} \int d^3p \, e^{-i\omega_{\boldsymbol{p}}t} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \tag{12}$$

where we used 3 and 7 to get the last line.

We can convert the integral to spherical coordinates in $\boldsymbol{p}$ space, so we have

$$\left\langle \boldsymbol{x} \left| e^{-iHt} \right| \psi \right\rangle = \frac{1}{(2\pi)^3} \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} dp \, d\theta \, d\phi \, p^2 \sin\theta e^{-i\omega_{\boldsymbol{p}}t} e^{ipr\cos\theta} \tag{13}$$

Here the symbol p in normal font indicates $p \equiv |\boldsymbol{p}|$ and not the four-momentum, and $r \equiv |\boldsymbol{x}|$. The integrals over ϕ and θ are fairly easy (or you can do them with Maple), and we end up with

$$\left\langle \boldsymbol{x} \left| e^{-iHt} \right| \psi \right\rangle = \frac{i}{4\pi^2 r} \int_0^{\infty} dp \, p \, e^{-i\omega_{\boldsymbol{p}}t} \left(e^{-ipr} - e^{ipr} \right) \tag{14}$$

The integrand is the product of two odd functions of p (p itself and $e^{-ipr} - e^{ipr}$) and an even function of p ($e^{-i\omega_{\boldsymbol{p}}t}$), so the entire integrand is an even function of p. We can therefore expand the range of integration so we have

$$\left\langle \boldsymbol{x}\left|e^{-iHt}\right|\psi\right\rangle = \frac{i}{8\pi^2 r}\int_{-\infty}^{\infty} dp\, p\, e^{-i\omega_{\boldsymbol{p}}t}\left(e^{-ipr}-e^{ipr}\right) \tag{15}$$

$$= \frac{i}{8\pi^2 r}\left[\int_{-\infty}^{\infty} dp\, p\, e^{-i\omega_{\boldsymbol{p}}t}e^{-ipr} - \right. \tag{16}$$

$$\left. \int_{-\infty}^{\infty} dp\, p\, e^{-i\omega_{\boldsymbol{p}}t}e^{ipr}\right] \tag{17}$$

Now by substituting $p \to -p$ in the first term, we have $dp \to -dp$, $p \to -p$, $e^{-ipr} \to e^{ipr}$ and the limits of the integral are reversed. We therefore have

$$\int_{-\infty}^{\infty} dp\, p\, e^{-i\omega_{\boldsymbol{p}}t}e^{-ipr} = \int_{\infty}^{-\infty} (-dp)\ (-p)\ e^{-i\omega_{\boldsymbol{p}}t}e^{ipr} \tag{18}$$

$$= -\int_{-\infty}^{\infty} dp\, p\, e^{-i\omega_{\boldsymbol{p}}t}e^{ipr} \tag{19}$$

Substituting this back into 17 we have

$$\left\langle \boldsymbol{x}\left|e^{-iHt}\right|\psi\right\rangle = -\frac{i}{4\pi^2 r}\int_{-\infty}^{\infty} dp\, p\, e^{-i\omega_{\boldsymbol{p}}t}e^{ipr} \tag{20}$$

with $\omega_{\boldsymbol{p}}$ given by 1. The integral, as far as I know, has no closed-form solution, but we're interested in seeing if it has a non-zero value outside the particle's light cone. On a space-time diagram, the light cone is given by the cone $r = t$. Events within the light cone have $t > r$, so we're interested in the value of $\left\langle \boldsymbol{x}\left|e^{-iHt}\right|\psi\right\rangle$ for the region $r > t$. Events in this region have a spacelike separation from the origin, and thus cannot be connected by any world line travelling slower than light. If we find that the probability for the particle to be found in this region is *not* zero, then the theory violates special relativity and cannot be valid.

We approach the integral in 20 by using contour integration. The idea is that we integrate the function along a closed contour in the complex plane, in which the momentum variable p becomes a complex variable:

$$p = x + iy \tag{21}$$

We write out the integral in full to get

$$\int_{-\infty}^{\infty} dp\, p\, e^{-i\omega_{\boldsymbol{p}}t}e^{ipr} = \int_{-\infty}^{\infty} dp\, p\, e^{ipr - i\sqrt{p^2+\mu^2}t} \tag{22}$$

We would like a contour that includes the entire real axis, since the desired integral is the integral along that portion of the contour. The problem is how to close the contour so we can apply Cauchy's theorem.

The difficulty arises with the square root. We can write the square root as

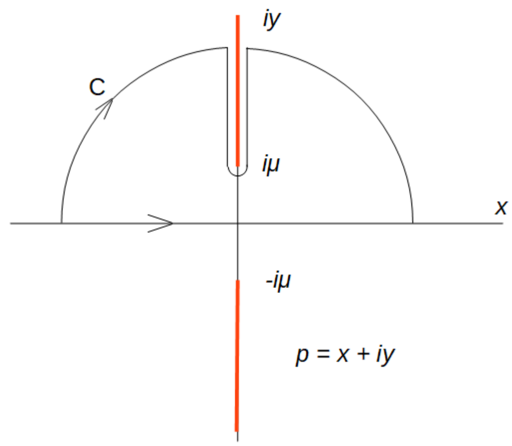

FIGURE 1. Contour for integrating 22.

$$\sqrt{p^2+\mu^2}=\sqrt{(p+i\mu)(p-i\mu)} \tag{23}$$

We now write each factor in modulus-argument form, so we have

$$\begin{aligned} p+i\mu &= ae^{i\alpha} \\ p-i\mu &= be^{i\beta} \end{aligned} \tag{24}$$

Consider the factor $p-i\mu$. This is zero at the point $p=i\mu$ (the lower end of the upper red line in Fig. 1). Now imagine we draw a circular path around the point $p=i\mu$, where the circle is small enough that it does *not* also contain the point $p=-i\mu$. Just to the right of the point $p=i\mu$, the square root of $p-i\mu$ has the value

$$(p-i\mu)^{1/2}=b^{1/2}e^{i\beta/2} \tag{25}$$

As we travel around the circle counterclockwise we will eventually traverse a full circle, at which point β has increased by 2π, so the square root is now

$$(p-i\mu)^{1/2}=b^{1/2}e^{i\beta/2}e^{i\pi}=-b^{1/2}e^{i\beta/2} \tag{26}$$

However, the angle α (the argument of the point $p+i\mu$) merely increases a bit and then decreases again, since the circle we're traversing doesn't encircle the point $p=-i\mu$. Thus the square root of the factor $p+i\mu$ doesn't change; it retains the same sign throughout.

Now if we draw a small circular path around the lower point $p=-i\mu$, then the argument of $(p+i\mu)^{1/2}=a^{1/2}e^{i\alpha/2}$ picks up a factor of π so the square root becomes $(p+i\mu)^{1/2}=a^{1/2}e^{i\alpha/2}e^{i\pi}=-a^{1/2}e^{i\alpha/2}$. However, traversing this small circle doesn't change the sign of the square root of $(p-i\mu)$.

That is, any path which encircles one of the points $p=\pm i\mu$ but not the other causes the sign of the overall square root 23 to change. However, if we choose a path that contains *both* the points $p=\pm i\mu$, then the signs of both of the square roots change, with the result that the product of the two remains the same sign.

The two points $p=\pm i\mu$ are thus the branch points of the square root term. A branch cut is therefore a line that prevents a contour from encircling exactly one of the branch points. We could define a branch cut as the line connecting the two points $p=\pm i\mu$, but if we did that, the branch cut would cross the x axis and interfere with the integration path in 22. The alternative is to draw branch cuts from each of $p=\pm i\mu$ out to infinity in directions away from the x axis. These branch cuts are shown in red in Fig. 1.

The contour that we will use for integration is therefore the contour C in the diagram. From Cauchy's theorem, the integral around this contour is equal to $2\pi i$ times the sum of the residues within the contour, but since the integrand in 22 has no singular points within the contour, the integral is zero. Therefore the integral along the x axis (in the positive direction) must be equal to the integral along the top part of the contour, which consists of a couple of arcs and the path around the branch cut. The direction along the top part of C is shown as clockwise, since the sum of the integral along the x axis and *counterclockwise* along C is zero, so the integral along the x axis is the *negative* of the counterclockwise integral along C, which is just the integral in the clockwise direction.

We need to convert the polar representations in 24 to $x+iy$ form. Since we're looking at the upper branch cut, it is only the sign of $\sqrt{p-i\mu}$ that changes from one side of the cut to the other. In the limit $x\to0^{+}$, that is, as we approach the cut from the right (positive) side of the x axis, we have

$$\sqrt{p^2+\mu^2} = \sqrt{(p+i\mu)(p-i\mu)} \tag{27}$$

$$\to \sqrt{(iy+i\mu)(iy-i\mu)} \tag{28}$$

$$= \sqrt{-y^2+\mu^2} \tag{29}$$

$$= \sqrt{-\left(y^2-\mu^2\right)} \tag{30}$$

$$= i\sqrt{y^2-\mu^2} \tag{31}$$

On the other side, the square root $\sqrt{p-i\mu}$ changes sign, but $\sqrt{p+i\mu}$ does not (see above) so

$$\sqrt{p^2+\mu^2} \to -i\sqrt{y^2-\mu^2} \tag{32}$$

We can now put all this together to get an idea of the behaviour of the integral. First, consider the parts of the contour that lie on the large arcs to either side of the branch cut. Here, we can write p in polar form as

$$p = Re^{i\theta} = R\cos\theta + iR\sin\theta \tag{33}$$

On the left-hand arc, $\frac{\pi}{2} < \theta < \pi$ so $\sin\theta > 0$. The real part of the exponents in the integrand in 22 is, in the limit of large R:

$$ipr - i\sqrt{p^2+\mu^2}t \to -rR\sin\theta + tR\sin\theta \tag{34}$$

We're interested in points outside the light cone, where $r > t$, so this quantity is negative, and the integrand in 22 therefore contains a negative exponent, so the integral will vanish for large R.

On the right-hand arc of C, $0 < \theta < \frac{\pi}{2}$, so $\sin\theta > 0$ again and the same argument follows, showing that this arc also contributes nothing to the integral.

The little arc around the bottom of the branch cut in Fig. 1 contributes an amount that is proportional to the arc length of this part of the contour. We're considering the limit as the contour approaches the branch cut, so this arc will diminish to zero radius, meaning that this also contributes nothing to the integral.

Thus we're left with the integrals along the two vertical sides of the branch cut. On the left side, we're travelling downwards from $y = +\infty$ to $y = \mu$, so, using 32, we have for this region:

$$\int_\infty^\mu dy\; y\; e^{-ry-\sqrt{y^2-\mu^2}t} = -\int_\mu^\infty dy\; y\; e^{-ry-\sqrt{y^2-\mu^2}t} \tag{35}$$

For the right side of the branch cut, we are travelling up from $y = \mu$ to $y = +\infty$, and we use 31, so we have now

$$\int_\mu^\infty dy\; y\; e^{-ry+\sqrt{y^2-\mu^2}t} \tag{36}$$

Combining all this into 20 gives us the final form of the integral:

$$\left\langle \boldsymbol{x} \middle| e^{-iHt} \middle| \psi \right\rangle = -\frac{i}{4\pi^2 r}\int_\mu^\infty dy\; y\; e^{-ry}\left(e^{\sqrt{y^2-\mu^2}t} - e^{-\sqrt{y^2-\mu^2}t}\right) \tag{37}$$

$$= -\frac{i}{2\pi^2 r}\int_\mu^\infty dy\; y\; e^{-ry}\sinh\sqrt{y^2-\mu^2}t \tag{38}$$

[Coleman seems to have dropped a minus sign between his equations 1.82 and 1.84. I can't see where this happened. It shouldn't matter since it is only the square modulus of the amplitude that is physically significant, but comments are welcome.]

The point of the result 38 is that, in the range of integration $[\mu,\infty]$ the integrand is strictly positive so the amplitude is definitely not zero. This means that if we use a hamiltonian given by 1, the theory can violate special relativity and, in particular, causality, since it is possible for two events on a particle's world line to be separated by a spacelike interval.

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