REAL KLEIN-GORDON FIELDS

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Post date: 20 August 2023.

When we derived the Klein-Gordon equation in field theory, we assumed that the field ϕ is complex, so that both the field itself and its complex conjugate $\phi^{\dagger}$ are viewed as independent fields. However, we can also take the field ϕ to be real, in which case there is only a single field. The Lagrangian in this case is the same as the one we used originally:

$$\mathcal{L}_0^0 = K\left(\partial_\alpha \phi \partial^\alpha \phi - \mu^2 \phi^2\right) \tag{1}$$

The superscript 0 on $\mathcal{L}_0^0$ indicates that we're dealing with a *scalar* field (as opposed to a field with spin) and the subscript 0 indicates that it's a *free* field (no potential terms). K is a constant. Written out in terms of time and space derivatives, this is

$$\mathcal{L}_0^0 = K\left(\dot{\phi}^2 - (\nabla\phi)\cdot(\nabla\phi) - \mu^2\phi^2\right) \tag{2}$$

In the complex field case, we defined the conjugate momentum density as

$$\pi_r \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}^r} \tag{3}$$

The complex field Lagrangian takes $K = 1$ and combines the field and its complex conjugate to give a real Lagrangian:

$$\mathcal{L}_0^0 = \dot{\phi}^{\dagger}\dot{\phi} - \nabla\phi^{\dagger}\cdot\nabla\phi - \mu^2\phi^{\dagger}\phi \tag{4}$$

This gives a conjugate momentum density of

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}^{\dagger} \tag{5}$$

In the real field case, taking $K = 1$ and using the same definition of conjugate momentum density gives $\pi = 2\dot{\phi}$. If we require the conjugate momentum density to have the same form 5 in the real field case (where $\dot{\phi}^{\dagger} = \dot{\phi}$ since ϕ is real), then we must take $K = \frac{1}{2}$, since from 2 we have

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = 2K\dot{\phi} = \dot{\phi} \tag{6}$$