

## RELATIVISTIC INVARIANT INTEGRATION MEASURE

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We look at a simple calculation here that is important because it's used many times in doing integrals in quantum field theory. In many cases, we need to do an integral over energy-momentum space of the form

$$\int d^4p f(p) \quad (1)$$

where  $f(p)$  is some function of the 4-momentum  $p$ .

First, we observe that the integration measure,  $d^4p$ , is Lorentz invariant. This is because, under a Lorentz transformation represented by the  $4 \times 4$  matrix  $\Lambda$ , the 4-momentum transforms as

$$p \rightarrow \Lambda p \quad (2)$$

This is a linear transformation (the elements of  $\Lambda$  depend only on things like the relative speed of two inertial frames, or, for a rotation, the angle of rotation, but not on the components of the 4-momentum  $p$ ). Therefore  $\Lambda$  is itself the Jacobian of the transformation. Thus the integration measure transforms according to

$$d^4p \rightarrow |\det \Lambda| d^4p \quad (3)$$

Given that (you can test this for the simple case of a boost in the  $x$  direction, but the conclusion is true generally)

$$\det \Lambda = \pm 1 \quad (4)$$

the measure  $d^4p$  remains invariant under a Lorentz transformation.

However, in many applications, we wish to restrict the integration over  $d^4p$  to the *mass shell*, that is, the set of 4-momenta that obey the energy-momentum relation

$$p^0 = \omega_{\mathbf{p}} = +\sqrt{|\mathbf{p}|^2 + \mu^2} \quad (5)$$

where  $\mathbf{p}$  is the 3-momentum and  $\mu$  is the mass of the particle. That is, the constraint becomes

$$p^2 = p_\nu p^\nu = \mu^2 \quad (6)$$

If we're given an integral of the form 1, we can enforce this constraint by using a delta function:

$$\int d^4 p f(p) = \int dp^0 [d^3 \mathbf{p} \delta(p^2 - \mu^2) \theta(p^0)] f(p) \quad (7)$$

The step function  $\theta(p^0)$  restricts  $p^0$  to  $p^0 > 0$ .

If we consider only the integration over  $p^0$ , we have

$$\int_{-\infty}^{\infty} dp^0 [d^3 \mathbf{p} \delta(p^2 - \mu^2) \theta(p^0)] f(p) \quad (8)$$

We can transform the delta function using the identity

$$\delta(g(x)) = \sum_i \frac{\delta(x - z_i)}{|g'(z_i)|} \quad (9)$$

where  $z_i$  are the zeroes of  $g(x)$ . Note in particular that  $g'(z_i)$  is the *derivative* of  $g$  evaluated at points where the original function  $g(x)$  (*not* the derivative!) is zero.

In this case,

$$p^2 - \mu^2 = (p^0)^2 - \mathbf{p}^2 - \mu^2 \quad (10)$$

$$= (p^0)^2 - \omega_{\mathbf{p}}^2 \quad (11)$$

so we have

$$g(p^0) = (p^0)^2 - \omega_{\mathbf{p}}^2 \quad (12)$$

$$g'(p^0) = 2p^0 \quad (13)$$

The function  $g(p^0)$  has zeroes at  $p^0 = \pm\omega_{\mathbf{p}}$ , but since the step function  $\theta(p^0)$  restricts  $p^0$  to  $p^0 > 0$ , we need consider only the zero at  $p^0 = \omega_{\mathbf{p}}$ . At this point  $g'(\omega_{\mathbf{p}}) = 2\omega_{\mathbf{p}}$ , so the formula 9 gives us

$$\delta(p^2 - \mu^2) = \frac{\delta(p^0 - \omega_{\mathbf{p}})}{2\omega_{\mathbf{p}}} \quad (14)$$

Inserting this into 8 gives us

$$\int_{-\infty}^{\infty} dp^0 [d^3\mathbf{p} \delta(p^2 - \mu^2) \theta(p^0)] f(p) = \int_0^{\infty} dp^0 \frac{\delta(p^0 - \omega_{\mathbf{p}})}{2\omega_{\mathbf{p}}} d^3\mathbf{p} f(p) \quad (15)$$

$$= \frac{d^3\mathbf{p}}{2\omega_{\mathbf{p}}} f(\mathbf{p}) \quad (16)$$

where all occurrences of  $p^0$  in the function  $f(p)$  have been replaced by  $\omega_{\mathbf{p}}$ , so that, from 5,  $f(p)$  becomes a function of the 3-momentum  $f(\mathbf{p})$ , since  $p^0$  is now written entirely in terms of  $\mathbf{p}$ .

The Lorentz invariant measure  $d^4p$  on the mass shell is thus equivalent to the measure

$$d^4p \rightarrow \frac{d^3\mathbf{p}}{2\omega_{\mathbf{p}}} \quad (17)$$

and it is this measure (not  $d^3\mathbf{p}$  on its own) that is Lorentz invariant.

This measure is used to define a normalization for a relativistic ket  $|p\rangle$ :

$$|p\rangle = \sqrt{(2\pi)^3} \sqrt{2\omega_{\mathbf{p}}} |\mathbf{p}\rangle \quad (18)$$

The usual unit operator

$$1 = \int d^3\mathbf{p} |\mathbf{p}\rangle \langle \mathbf{p}| \quad (19)$$

can therefore be written

$$1 = \frac{1}{(2\pi)^3} \int d^4p \delta(p^2 - \mu^2) \theta(p^0) |p\rangle \langle p| \quad (20)$$

$$= \frac{1}{(2\pi)^3} \int \frac{d^3\mathbf{p}}{2\omega_{\mathbf{p}}} (2\pi)^3 2\omega_{\mathbf{p}} |\mathbf{p}\rangle \langle \mathbf{p}| \quad (21)$$

$$= \int d^3\mathbf{p} |\mathbf{p}\rangle \langle \mathbf{p}| \quad (22)$$

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