ROTATION SYMMETRY AND CONSERVATION OF ANGULAR MOMENTUM

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 8 August 2021.

Coleman's third example of applying Noether's theorem is with symmetry under spatial rotation. We consider the Lagrangian

$$L=\frac{1}{2}\sum_{r}m_{r}\dot{\boldsymbol{x}}^{r}\cdot\dot{\boldsymbol{x}}^{r}+\sum_{r>s}V^{(r,s)}\left(\left|\boldsymbol{x}^{r}-\boldsymbol{x}^{s}\right|\right) \tag{1}$$

This describes a system with the usual classical kinetic energy in the first term and a potential energy $V^{(r,s)}$ that depends only on the distances between the particles, and not on their absolute positions.

We consider a fixed rotation about an axis $\boldsymbol{e}$ through an angle λ. In Coleman's example, the axis $\boldsymbol{e}$ can point in any direction. Working out the details of such a rotation is a lengthy and rather messy derivation, but the result is that we can represent such a rotation in matrix form by using the Rodrigues rotation formula. The derivation of this formula would take us too far afield from the physics of this example, but if you're interested, a derivation is given on the Wikipedia page. The result is as follows.

For a vector $\boldsymbol{x}^{r}$ (where r is an index identifying the vector; it's not a component of the vector), under rotation it becomes

$$\boldsymbol{x}^{r}\rightarrow R(\lambda,\boldsymbol{e})\,\boldsymbol{x}^{r} \tag{2}$$

where the rotation matrix R is given by

$$R(\lambda,\boldsymbol{e})=I+(\sin\lambda)\,K+(1-\cos\lambda)\,K^{2} \tag{3}$$

with I being the 3×3 identity matrix, and the matrix K defined by

$$K\equiv\left[\begin{array}{ccc}0 & -e_{z} & e_{y}\\ e_{z} & 0 & -e_{x}\\ -e_{y} & e_{x} & 0\end{array}\right] \tag{4}$$

$$K^{2}=\left[\begin{array}{ccc}-e_{y}^{2}-e_{z}^{2} & e_{y}e_{x} & e_{z}e_{x}\\ e_{y}e_{x} & -e_{x}^{2}-e_{z}^{2} & e_{z}e_{y}\\ e_{z}e_{x} & e_{z}e_{y} & -e_{x}^{2}-e_{y}^{2}\end{array}\right] \tag{5}$$

Written as a vector equation, this is

$$\boldsymbol{x}^r \to \boldsymbol{x}^r \cos\lambda + (\boldsymbol{e}\times\boldsymbol{x}^r)\sin\lambda + \boldsymbol{e}(\boldsymbol{e}\cdot\boldsymbol{x}^r)(1-\cos\lambda) \tag{6}$$

To apply Noether's theorem to this symmetry, we need to find

$$DL = \frac{\partial L}{\partial q^a} Dq^a + \frac{\partial L}{\partial \dot{q}^a} D\dot{q}^a \tag{7}$$

$$= \frac{\partial L}{\partial q^a} Dq^a + p_a D\dot{q}^a \tag{8}$$

where

$$p_a \equiv \frac{\partial L}{\partial \dot{q}^a} \tag{9}$$

is the momentum conjugate to the coordinate q^a, and

$$Dq^a \equiv \left.\frac{\partial q^a}{\partial \lambda}\right|_{\lambda=0} \tag{10}$$

$$D\dot{q}^a = \frac{d}{dt} Dq^a \tag{11}$$

In this example, we can calculate Dq^a from 6

$$Dq^a = D\boldsymbol{x}^r \tag{12}$$

$$= \left.\frac{\partial \boldsymbol{x}^r}{\partial \lambda}\right|_{\lambda=0} \tag{13}$$

$$= \left.-\boldsymbol{x}^r \sin\lambda + (\boldsymbol{e}\times\boldsymbol{x}^r)\cos\lambda - \boldsymbol{e}(\boldsymbol{e}\cdot\boldsymbol{x}^r)\sin\lambda\right|_{\lambda=0} \tag{14}$$

$$= \boldsymbol{e}\times\boldsymbol{x}^r \tag{15}$$

To calculate DL in 8 we could write out the Lagrangian in terms of the coordinates of $\boldsymbol{x}^r$, as we did in detail for translation symmetry and then calculate all the derivatives. However, it's easier to use the physical argument that under a rigid rotation of the coordinate system, none of the particles' speeds will change, and neither will the distances between the particles. Thus the Lagrangian 1 will not change under a rotation, so we can assert that

$$DL = 0 \tag{16}$$

This symmetry results in a conservation law if

$$DL = \frac{dF}{dt} \tag{17}$$

for some function $F(q^a, \dot{q}^a, t)$. Since $DL = 0$, we have

$$F = 0 \tag{18}$$

The conserved quantity is Q, defined by

$$Q = p_a Dq^a - F \tag{19}$$

so from 19 we can now find the conserved quantity. As Dq^a is a vector in this case (from 15) and p_a is also the component of the momentum vector, we have (with an implied sum over r):

$$Q = p_a Dq^a = \boldsymbol{p}_r \cdot (\boldsymbol{e} \times \boldsymbol{x}^r) \tag{20}$$

From linear algebra, we know that the vectors in a scalar triple product such as Q can be cyclically permuted, so this is equivalent to

$$Q = \boldsymbol{e} \cdot (\boldsymbol{x}^r \times \boldsymbol{p}_r) \tag{21}$$

$$= \boldsymbol{e} \cdot \boldsymbol{J} \tag{22}$$

where

$$\boldsymbol{J} = \boldsymbol{x}^r \times \boldsymbol{p}_r \tag{23}$$

is the total angular momentum.

As in the translation case, the vector $\boldsymbol{e}$ is arbitrary, so we have an infinite number of conservation laws, each stating that the component of $\boldsymbol{J}$ along $\boldsymbol{e}$ is conserved. In three dimensions, we can choose $\boldsymbol{e}$ to be each of the 3 coordinate axes and obtain 3 conservation laws, one for each component of angular momentum.