SCALAR FIELDS - SUMMARY

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Before jumping into Feynman diagrams, it's worthwhile summarizing the steps we have taken so far in constructing a quantum field theory of scalar fields, so we can take a bird's eye view of the theory so far. If you're unfamiliar with any of these steps, follow the link back to the page explaining it.

We began with five requirements that the quantum field must satisfy. These are

(1) The fields must be linear combinations of creation and annihilation operators.
(2) The fields must be hermitian, since they are observable.
(3) They must transform properly under translations.
(4) They must transform properly under Lorentz transformations.
(5) Fields separated by a spacelike interval must commute (to avoid violating relativity).

Applying these conditions leads to the final form for a single free scalar quantum field:

$$\phi(x)=\int\frac{d^{3}\boldsymbol{p}}{\sqrt{(2\pi)^{3}}\sqrt{2\omega_{\boldsymbol{p}}}}\left[a_{\boldsymbol{p}}e^{-ip\cdot x}+a_{\boldsymbol{p}}^{\dagger}e^{ip\cdot x}\right] \tag{1}$$

The Lagrangian for a single free scalar field is

$$\mathscr{L}=\frac{1}{2}\left(\partial^{\mu}\phi\partial_{\mu}\phi-\mu^{2}\phi^{2}\right) \tag{2}$$

where μ is the mass of the particle.

By considering a system of two scalar fields ϕ^1 and ϕ^2, we can combine them in the forms:

$$\psi=\frac{1}{\sqrt{2}}\left(\phi^{1}+i\phi^{2}\right) \tag{3}$$

$$\psi^{*}=\frac{1}{\sqrt{2}}\left(\phi^{1}-i\phi^{2}\right) \tag{4}$$

and get expressions for these fields in terms of creation and annihilation operators:

$$\begin{aligned}\psi(x) &= \int \frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3}\sqrt{2\omega_{\boldsymbol{p}}}}\left(b_{\boldsymbol{p}}e^{-ip\cdot x}+c_{\boldsymbol{p}}^{\dagger}e^{ip\cdot x}\right)\\ \psi^{\dagger}(x) &= \int \frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3}\sqrt{2\omega_{\boldsymbol{p}}}}\left(b_{\boldsymbol{p}}^{\dagger}e^{ip\cdot x}+c_{\boldsymbol{p}}e^{-ip\cdot x}\right)\end{aligned} \tag{5}$$

(Sometimes $\psi^{\dagger}$ is written as ψ^{*}.) Here, the operators $b_{\boldsymbol{p}}$ and $b_{\boldsymbol{p}}^{\dagger}$ annihilate and create a nucleon N, while $c_{\boldsymbol{p}}$ and $c_{\boldsymbol{p}}^{\dagger}$ annihilate and create an antinucleon $\overline{N}$.

This system has a Lagrangian

$$\mathscr{L} = \partial_{\mu}\psi^{\dagger}\partial^{\mu}\psi - m^2\psi^{\dagger}\psi \tag{6}$$

This is as far as we can go in discussing free scalar fields. We now need to consider adding an interaction to the fields. Since most interactions result in systems that have no exact solution, we need to use perturbation theory. To do this, we use the interaction picture, in which the hamiltonian is given as the sum of a free time-independent term $H_0(p,q)$ and an interaction term $H'(p,q,t)$ that could be time-dependent.The advantage of using the interaction picture is that the operators are now independent of the interaction, with all the effects of the interaction being shifted into the quantum states $|\psi\rangle$. That is, operators (including the field operators above) are those of free fields, so they have the forms given above, even in the presence of the interaction.

The evolution of the states $|\psi(t)\rangle_I$ is given by the formula

$$|\psi(t)\rangle_I = U_I(t,t')\,|\psi(t')\rangle_I \tag{7}$$

where U_I is the evolution operator in the interaction picture. To find U_I we must solve the differential equation

$$i\frac{\partial}{\partial t}U_I(t,t') = H_I(t)\,U_I(t,t') \tag{8}$$

where $H_I(t)$ is the interaction term in the hamiltonian, given by

$$H_I(t) = H'(p_I,q_I,t) \tag{9}$$

where the momentum p_I and position q_I are given by

$$\begin{aligned} p_I(t) &= e^{iH_0 t} p_S e^{-iH_0 t} \\ q_I(t) &= e^{iH_0 t} q_S e^{-iH_0 t} \end{aligned} \tag{10}$$

where the subscript S indicates the Schrödinger picture, where operators are independent of time.

The solution of 8 is given formally by Dyson's formula:

$$U_I(t,t') = T\exp\left(-i\int_{t'}^{t} dt'' H_I(t'')\right) \tag{11}$$

where T indicates time-ordering.

In practice, the exponential in 11 is expanded in a power series, and we evaluate only the first few terms in the series. The evaluation uses Wick's theorem, which converts a time-ordered product into a sum of normal-ordered products and contractions. A contraction of two fields is defined as

$$\overbrace{A(x)\,B}(y) \equiv T(A(x)\,B(y)) - :A(x)\,B(y): \tag{12}$$

and turns out to be just a number, given by

$$\overbrace{A(x)\,B}(y) = \begin{cases} \Delta_+(x-y) & \text{if } A = B \\ 0 & \text{if } A \neq B \end{cases} \tag{13}$$

where

$$\Delta_+(x-y) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3 (2\omega_{\boldsymbol{p}})} e^{-ip\cdot(x-y)} \tag{14}$$

For the two types of fields described above in 1 and 5, the contractions (that is, $\Delta_+(x-y)$) come out to

$$\overbrace{\phi(x)\,\phi}(y) = \lim_{\epsilon\to 0^+}\int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{i}{p^2-\mu^2+i\epsilon} \tag{15}$$

$$\overbrace{\psi^\dagger(x)\,\psi}(y) = \lim_{\epsilon\to 0^+}\int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{i}{p^2-m^2+i\epsilon} \tag{16}$$

A contraction of two fields of different types (such as ψ with ϕ), or ψ with ψ or $\psi^\dagger$ with $\psi^\dagger$ always gives zero.

In the general case, then, when the interaction is given as a term involving a product of fields, such $\psi^\dagger\psi\phi$, the expansion of 11 will involve terms containing time-ordered powers of this product, such as $T\left(\psi^\dagger(x)\,\psi(x)\,\phi(x)\,\psi^\dagger(y)\,\psi(y)\,\phi(y)\right)$. Each such term can be converted to a sum of normal-ordered products and contractions using Wick's theorem. The contractions can be evaluated using their integral forms, leaving us with a normal-ordered product. If we sandwich a normal-ordered product between two vacuum states, we always get zero, since an annihilation operator acting on the vacuum ket $|0\rangle$ gives zero, and a creation operator acting back on a bra $\langle 0|$ gives zero. Therefore, from 12 we have

$$\langle 0|T(A(x)B(y))|0\rangle = \left\langle 0\left|\contraction{}{A}{(x)}{B}A(x)B(y)\right|0\right\rangle \tag{17}$$

$$= \contraction{}{A}{(x)}{B}A(x)B(y) \tag{18}$$

where the last step follows since the contraction is just a number and can be taken outside the matrix element.

Pingbacks

Pingback: Wick diagrams to Feynman diagrams