SOLUTIONS TO THE 4D WAVE EQUATION

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The wave equation in 4d, using the Lorenz gauge, is

$$\partial^{\alpha}\partial_{\alpha}A^{\mu}(x)=0 \tag{1}$$

where $A^{\mu}(x)$ is the 4d vector potential, which is a function of spacetime x. A solution to this equation is given by

$$A^{\mu}(x)=\sum_{r,\boldsymbol{k}}\frac{1}{\sqrt{2V\omega_{\boldsymbol{k}}}}\varepsilon_{r}^{\mu}A_{r}(\boldsymbol{k})e^{-ikx}+\sum_{r,\boldsymbol{k}}\frac{1}{\sqrt{2V\omega_{\boldsymbol{k}}}}\varepsilon_{r}^{\mu}B_{r}^{\dagger}(\boldsymbol{k})e^{ikx} \tag{2}$$

Here, $A_{r}(\boldsymbol{k})$ and $B_{r}^{\dagger}(\boldsymbol{k})$ are (possibly complex) numerical coefficients. The ε_{r}^{μ} are the polarization vectors, which we'll discuss in a minute. This solution is given for a finite volume V, so the values of k are discrete rather than continuous, as they would be in infinite space.

We can show that 2 is a solution by direct substitution. We have for the first term, since the only term that involves x is the exponential:

$$\partial^{\alpha}\partial_{\alpha}\left[\sum_{r,\boldsymbol{k}}\frac{1}{\sqrt{2V\omega_{\boldsymbol{k}}}}\varepsilon_{r}^{\mu}A_{r}(\boldsymbol{k})e^{-ikx}\right]=(-i)^{2}\sum_{r,\boldsymbol{k}}\frac{1}{\sqrt{2V\omega_{\boldsymbol{k}}}}\varepsilon_{r}^{\mu}A_{r}(\boldsymbol{k})k^{\alpha}k_{\alpha}e^{-ikx} \tag{3}$$

For photons

$$k^{\alpha}k_{\alpha}=\omega_{\boldsymbol{k}}^{2}-\boldsymbol{p}^{2}=0 \tag{4}$$

so we see that the RHS of 3 is zero. The second term in 2 is zero by the same argument:

$$\partial^{\alpha}\partial_{\alpha}\left[\sum_{r,\boldsymbol{k}}\frac{1}{\sqrt{2V\omega_{\boldsymbol{k}}}}\varepsilon_{r}^{\mu}B_{r}^{\dagger}(\boldsymbol{k})e^{ikx}\right]=i^{2}\sum_{r,\boldsymbol{k}}\frac{1}{\sqrt{2V\omega_{\boldsymbol{k}}}}\varepsilon_{r}^{\mu}B_{r}^{\dagger}(\boldsymbol{k})k^{\alpha}k_{\alpha}e^{ikx}=0 \tag{5}$$

The wave equation 1 is similar to the Klein-Gordon equation for a scalar field, which is

$$\left(\partial^{\alpha}\partial_{\alpha}+m^{2}\right)\phi=0 \tag{6}$$

The general plane wave solution to the Klein-Gordon equation gave a complex field ϕ. For the current equation 1, we can observe some key points.

(1) The electromagnetic wave equation is equivalent to the Klein-Gordon equation with $m=0$. This makes sense, since the fundamental particle of electromagnetism is the photon, which has zero mass.
(2) Since the 4-potential A^{μ} is used to calculate the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$, which are observables, it must be real rather than complex.
(3) Since the 4-potential A^{μ} is a 4-vector rather than a scalar, its solution must also be a 4-vector.

Applying point 2 to 2, and assuming that ε_{r}^{μ} is real, we see that the second term must be the complex conjugate of the first term, which means that

$$B_{r}^{\dagger}(\boldsymbol{k})=A_{r}^{\dagger}(\boldsymbol{k}) \tag{7}$$

Thus the solution 2 becomes

$$A^{\mu}(x)=\sum_{r,\boldsymbol{k}}\frac{1}{\sqrt{2V\omega_{\boldsymbol{k}}}}\varepsilon_{r}^{\mu}A_{r}(\boldsymbol{k})e^{-ikx}+\sum_{r,\boldsymbol{k}}\frac{1}{\sqrt{2V\omega_{\boldsymbol{k}}}}\varepsilon_{r}^{\mu}A_{r}^{\dagger}(\boldsymbol{k})e^{ikx} \tag{8}$$

Considering point 3 above, we can see that the requirement that the solution be a 4-vector rather than a scalar is fulfilled by the polarization vectors ε_{r}^{μ}. Since we're working in 4d space, the polarization vectors each have 4 components, and there must be 4 independent vectors in order that we can span the 4d space.

The subscript r indicates which vector we're talking about, with $r=0,1,2,3$. The superscript μ indicates which component of the vector ε_{r} we're considering. Thus $\mu=0,1,2,3$ as well. It's important not to get these two indexes confused: remember that r labels an entire 4d vector, while μ labels a specific component of a 4d vector.

At this stage, all that is required of the ε_{r}s is that they are independent and span the space. Thus one choice is

$$\begin{aligned}
\varepsilon_{0}&=\left(\begin{array}{cccc}1&0&0&0\end{array}\right)^{T}\\
\varepsilon_{1}&=\left(\begin{array}{cccc}0&1&0&0\end{array}\right)^{T}\\
\varepsilon_{2}&=\left(\begin{array}{cccc}0&0&1&0\end{array}\right)^{T}\\
\varepsilon_{3}&=\left(\begin{array}{cccc}0&0&0&1\end{array}\right)^{T}
\end{aligned} \tag{9}$$

where the T indicates the transpose, so the ε_{r}s are column vectors.

With this choice, the polarization vectors are orthogonal, so that

$$\varepsilon_{\mu r}\varepsilon_{s}^{\mu}=0 \quad \text{if } r\neq s \tag{10}$$

We also have

$$\begin{aligned}
\varepsilon_{\mu 0}\varepsilon_{0}^{\mu} &= g_{\mu\nu}\varepsilon_{0}^{\mu}\varepsilon_{0}^{\nu}=+1\\
\varepsilon_{\mu 1}\varepsilon_{1}^{\mu} &= g_{\mu\nu}\varepsilon_{1}^{\mu}\varepsilon_{1}^{\nu}=-1\\
\varepsilon_{\mu 2}\varepsilon_{2}^{\mu} &= g_{\mu\nu}\varepsilon_{2}^{\mu}\varepsilon_{2}^{\nu}=-1\\
\varepsilon_{\mu 3}\varepsilon_{3}^{\mu} &= g_{\mu\nu}\varepsilon_{3}^{\mu}\varepsilon_{3}^{\nu}=-1
\end{aligned} \tag{11}$$

Here the metric tensor $g_{\mu\nu}$ has the signature $g_{00}=+1$, $g_{ii}=-1$.

These conditions are often denoted by

$$\varepsilon_{\mu r}\varepsilon_{s}^{\mu}=-\zeta_{r}\delta_{rs} \tag{12}$$

where

$$\begin{aligned}
\zeta_{0} &= -1\\
\zeta_{i} &= +1
\end{aligned} \tag{13}$$

Other choices for the polarization vectors are used, but we'll leave those to a future post.

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