THE S OPERATOR AS A TIME-ORDERED EXPONENTIAL

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Post date: 21 November 2023.

In the interaction picture (IP), the evolution operator $U_I(t,t')$ (using Coleman's notation) satisfies the differential equation

$$i\frac{\partial}{\partial t}U_I\left(t,t'\right)=H_I(t)\,U_I\left(t,t'\right) \tag{1}$$

In what follows, we'll be examining Klauber's derivations, so we will restate 1 using Klauber's notation. To get there, we start with the equation of motion for IP states

$$i\frac{d}{dt}\left|\mathbf{\Psi}\right\rangle_I=H_I^I\left|\mathbf{\Psi}\right\rangle_I \tag{2}$$

where H_I^I is the interaction portion of the Hamiltonian in the IP. Recall that, for operators, a superscript I means that we're using the IP, and a subscript I means we're looking at the portion of the operator due to interactions.

We now define the S operator $S_{oper}(t,t_i)$ to be the operator that evolves a state from an initial time t_i to a different time t. That is, still in the IP:

$$\left|\mathbf{\Psi}(t)\right\rangle_I=S_{oper}(t,t_i)\left|\mathbf{\Psi}(t_i)\right\rangle_I \tag{3}$$

Note that $\left|\mathbf{\Psi}(t_i)\right\rangle_I$ is the state at the initial time t_i and, as such, does not depend on the general time variable t.

Inserting this into 2 we have

$$i\frac{d}{dt}\left(S_{oper}(t,t_i)\left|\mathbf{\Psi}(t_i)\right\rangle_I\right)=H_I^IS_{oper}(t,t_i)\left|\mathbf{\Psi}(t_i)\right\rangle_I \tag{4}$$

Carrying out the derivative and remembering that $\left|\mathbf{\Psi}(t_i)\right\rangle_I$ doesn't depend on t, we have

$$i\frac{dS_{oper}(t,t_i)}{dt}\left|\mathbf{\Psi}(t_i)\right\rangle_I=H_I^IS_{oper}(t,t_i)\left|\mathbf{\Psi}(t_i)\right\rangle_I \tag{5}$$

This equation must be true for all initial times t_i, so we must have

$$i\frac{dS_{oper}(t,t_i)}{dt}=H_I^IS_{oper}(t,t_i) \tag{6}$$

In general, the interaction Hamiltonian H_I^I depends on time, so we can write a formal solution as

$$S_{oper}(t,t_i) = \exp\left[-i\int_{t_i}^{t}\mathcal{H}_I^I d^4x\right] \tag{7}$$

where we've used the Hamiltonian density $\mathcal{H}_I^I$ and integrated over the volume V containing the system, as well as over time from t_i to t. In terms of the full Hamiltonian, this is

$$S_{oper}\left(t_f,t_i\right) = \exp\left[-i\int_{t_i}^{t_f} H_I^I dt\right] \tag{8}$$

where we now have a single integral between the initial time t_i and the final time t_f.

To actually evaluate 8, we need to expand the exponential as a power series, in the form

$$\exp\left[-i\int_{t_i}^{t_f} H_I^I dt\right] = I - i\int_{t_i}^{t_f} H_I^I dt + \frac{(-i)^2}{2!}\int_{t_i}^{t_f} H_I^I dt_1 \int_{t_i}^{t_f} H_I^I dt_2 + \ldots \tag{9}$$

The catch with this expansion is that it is to be applied to some initial state at time t_i, so the integrations in the second and higher order terms need to be done with earlier times before later times. A full discussion was given earlier of what is known as Dyson's time ordering solution. That is, we have

$$S_{oper}\left(t_f,t_i\right) = T\exp\left[-i\int_{t_i}^{t_f} H_I^I dt\right] \tag{10}$$

where T is the time-ordering operator, which arranges terms so that earlier times are always to the right of later times.

Trying to do the integrations using Dyson's formula is very difficult, however. It is much easier to transform the time ordering to normal ordering using Wick's theorem.

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