

## TIME EVOLUTION OF CREATION AND ANNIHILATION OPERATORS

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The Heisenberg equation of motion for an operator  $Q$  is

$$\frac{dQ}{dt} = -\frac{i}{\hbar} [Q, H] \quad (1)$$

where  $H$  is the Hamiltonian.

For a for a real scalar field  $\phi$ , we can write the Hamiltonian in terms of creation and annihilation operators as

$$H = \sum_k E_k a_k^\dagger a_k \quad (2)$$

where these operators satisfy the usual commutation relations

$$[a_k, a_j^\dagger] = \delta_{kj} \quad (3)$$

$$[a_k, a_j] = [a_k^\dagger, a_j^\dagger] = 0 \quad (4)$$

Applying the equation of motion we have

$$\frac{da_k^\dagger(t)}{dt} = -\frac{i}{\hbar} [a_k^\dagger, H] \quad (5)$$

$$= -\frac{i}{\hbar} \sum_j E_j a_j^\dagger [a_k^\dagger, a_j] \quad (6)$$

$$= +\frac{i}{\hbar} \sum_j E_j a_j^\dagger [a_j, a_k^\dagger] \quad (7)$$

$$= \frac{i}{\hbar} \sum_j E_j a_j^\dagger \delta_{jk} \quad (8)$$

$$= \frac{i}{\hbar} E_k a_k^\dagger(t) \quad (9)$$

This differential equation has the formal solution

$$a_k^\dagger(t) = a_k^\dagger(0) e^{iE_k t/\hbar} \quad (10)$$

The corresponding equation for  $a_k(t)$  can be obtained by a similar procedure, or by just taking the conjugate of 10:

$$a_k(t) = a_k(0) e^{-iE_k t/\hbar} \quad (11)$$

Now suppose we have a more general operator defined by

$$\hat{X} = X_{\ell m} a_\ell^\dagger a_m \quad (12)$$

where the  $X_{\ell m}$  are just numbers (independent of time) and (I assume) there is no implied sum over  $\ell$  and  $m$ . To find the time dependence of  $\hat{X}$  we have (I'll drop the hat from  $X$  but it's understood to be an operator from here on):

$$\frac{dX}{dt} = -\frac{i}{\hbar} [X, H] \quad (13)$$

$$= -\frac{i}{\hbar} X_{\ell m} \sum_k E_k [a_\ell^\dagger a_m, a_k^\dagger a_k] \quad (14)$$

We'll now work out the commutator in the sum, using 4

$$[a_\ell^\dagger a_m, a_k^\dagger a_k] = a_\ell^\dagger a_m a_k^\dagger a_k - a_k^\dagger a_k a_\ell^\dagger a_m \quad (15)$$

$$= a_\ell^\dagger a_k^\dagger a_m a_k + a_\ell^\dagger a_k \delta_{km} - a_k^\dagger a_\ell^\dagger a_k a_m - a_k^\dagger a_m \delta_{kl} \quad (16)$$

$$= a_\ell^\dagger a_k^\dagger a_m a_k - a_\ell^\dagger a_k^\dagger a_m a_k + a_\ell^\dagger a_k \delta_{km} - a_k^\dagger a_m \delta_{kl} \quad (17)$$

$$= a_\ell^\dagger a_k \delta_{km} - a_k^\dagger a_m \delta_{kl} \quad (18)$$

Inserting this back into 14 we have

$$\frac{dX(t)}{dt} = -\frac{i}{\hbar} X_{\ell m} \sum_k E_k (a_\ell^\dagger a_k \delta_{km} - a_k^\dagger a_m \delta_{kl}) \quad (19)$$

$$= -\frac{i}{\hbar} X_{\ell m} (E_m a_\ell^\dagger a_m - E_\ell a_\ell^\dagger a_m) \quad (20)$$

$$= -\frac{i}{\hbar} X_{\ell m} a_\ell^\dagger a_m (E_m - E_\ell) \quad (21)$$

$$= -\frac{i}{\hbar} (E_m - E_\ell) X(t) \quad (22)$$

We can again integrate this to give

$$X(t) = X(0) e^{i(E_\ell - E_m)t/\hbar} \quad (23)$$

We could have guessed this result by looking at 10 and 11, since in 12, the operator  $a_\ell^\dagger$  evolves according to 10 and  $a_m$  evolves according to 11.