TIME EVOLUTION OPERATOR

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In the Schrödinger picture (where the time dependence of a quantum system resides in the wave functions rather than the operators), we can write the time evolution of a wave function as

$$\psi(t_2) = U(t_2, t_1)\psi(t_1) \tag{1}$$

where the *time evolution operator* $U(t_2,t_1)$ operates on a wave function $\psi(t_1)$ at time t_1 and 'evolves' it to the wave function $\psi(t_2)$ at time t_2.

Lancaster & Blundell denote an operator by putting a hat over it, as in $\hat{U}(t_2,t_1)$, but to save typing I'll omit the hat since it's fairly obvious from the context what the operators are.

L&B list five properties that a time evolution operator should have. One of these properties is that it must satisfy the Schrödinger equation, as in

$$i\frac{d}{dt_2}U(t_2,t_1) = HU(t_2,t_1) \tag{2}$$

where H is the Hamiltonian operator, assumed not to depend on time. This can be formally integrated to give

$$U(t_2,t_1) = e^{-iH(t_2-t_1)} \tag{3}$$

We can now demonstrate that this form does satisfy the five required properties.

Property 1. $U(t_1,t_1) = 1$. From 3 with $t_2 = t_1$ we have

$$U(t_1,t_1) = e^{-iH(t_1-t_1)} = e^0 = 1 \tag{4}$$

Property 2. $U(t_3,t_2)U(t_2,t_1) = U(t_3,t_1)$ (the composition law). We have

$$U(t_3,t_2)U(t_2,t_1) = e^{-iH(t_3-t_2)}e^{-iH(t_2-t_1)} \tag{5}$$

$$= e^{-iH(t_3-t_1)} \tag{6}$$

$$= U(t_3,t_1) \tag{7}$$

Property 3. This is just the differential equation 2 from which 3 was derived, so it's automatically satisfied.

Property 4. $U(t_1,t_2) = U^{-1}(t_2,t_1)$ (the inverse property). We have

$$U(t_1,t_2)U(t_2,t_1) = e^{-iH(t_1-t_2)}e^{-iH(t_2-t_1)} \tag{8}$$
$$= 1 \tag{9}$$

Therefore $U(t_1,t_2) = U^{-1}(t_2,t_1)$. Taking the inverse of a time evolution operator evolves the state backwards in time.

Property 5. $U^{\dagger}(t_2,t_1)U(t_2,t_1) = 1$ (the unitary property). In other words, the hermitian conjugate of U is also its inverse. We have

$$U^{\dagger}(t_2,t_1) = e^{iH^{\dagger}(t_2-t_1)} \tag{10}$$
$$= e^{iH(t_2-t_1)} \tag{11}$$
$$= U^{-1}(t_2,t_1) \tag{12}$$

The second line follows because the Hamiltonian is hermitian, so $H^{\dagger} = H$.

Thus all five properties are satisfied.