TIME TRANSLATION SYMMETRY AND CONSERVATION OF ENERGY

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Coleman's second example of applying Noether's theorem is with symmetry under time translation. This time, we consider a general Lagrangian

$$L = L(q^a, \dot{q}^a) \tag{1}$$

The only assumption here is that L does not depend explicitly on time.

We consider the transformation

$$q^a \to q^a(t+\lambda) \tag{2}$$

That is, we translate the time coordinate t by a fixed amount λ for all generalized coordinates.

To apply Noether's theorem to this symmetry, we need to find

$$DL = \frac{\partial L}{\partial q^a} Dq^a + \frac{\partial L}{\partial \dot{q}^a} D\dot{q}^a \tag{3}$$

$$= \frac{\partial L}{\partial q^a} Dq^a + p_a D\dot{q}^a \tag{4}$$

where

$$p_a \equiv \frac{\partial L}{\partial \dot{q}^a} \tag{5}$$

is the momentum conjugate to the coordinate q^a, and

$$Dq^a \equiv \left.\frac{\partial q^a}{\partial \lambda}\right|_{\lambda=0} \tag{6}$$

$$D\dot{q}^a = \frac{d}{dt} Dq^a \tag{7}$$

In this example,

$$Dq^a = \left.\frac{\partial q^a}{\partial\lambda}\right|_{\lambda=0} \tag{8}$$

$$= \frac{\partial q^a}{\partial t} = \dot{q}^a \tag{9}$$

The last line follows from 2 because t and λ always appear in the combination $t+\lambda$, so a derivative with respect to one of t and λ is the same as the derivative with respect to other.

To calculate DL in 4 we need the following:

$$\frac{\partial L}{\partial q^a} Dq^a = \frac{\partial L}{\partial q^a}\dot{q}^a \tag{10}$$

$$D\dot{q}^a = \frac{\partial}{\partial t} Dq^a \tag{11}$$

$$= \ddot{q}^a \tag{12}$$

so we have

$$DL = \frac{\partial L}{\partial q^a} Dq^a + \frac{\partial L}{\partial \dot{q}^a} D\dot{q}^a \tag{13}$$

$$= \frac{\partial L}{\partial q^a}\dot{q}^a + \frac{\partial L}{\partial \dot{q}^a}\ddot{q}^a \tag{14}$$

The total derivative of a general Lagrangian $L(q^a, \dot{q}^a, t)$ with respect to time is, from the chain rule:

$$\frac{dL}{dt} = \frac{\partial L}{\partial q^a}\dot{q}^a + \frac{\partial L}{\partial \dot{q}^a}\ddot{q}^a + \frac{\partial L}{\partial t} \tag{15}$$

Therefore, we have

$$DL = \frac{\partial L}{\partial q^a}\dot{q}^a + \frac{\partial L}{\partial \dot{q}^a}\ddot{q}^a \tag{16}$$

$$= \frac{dL}{dt} - \frac{\partial L}{\partial t} \tag{17}$$

However, the Lagrangian we're considering here has no explicit time dependence, so

$$\frac{\partial L}{\partial t} = 0 \tag{18}$$

and therefore

$$DL = \frac{dL}{dt} \tag{19}$$

This symmetry results in a conservation law if

$$DL = \frac{dF}{dt} \tag{20}$$

for some function $F(q^a, \dot{q}^a, t)$.

From 19, we see that

$$F = L \tag{21}$$

The conserved quantity is Q, defined by

$$Q = p_a D q^a - F \tag{22}$$

so from 22 we can now find the conserved quantity, using 9:

$$Q = p_a D q^a - F \tag{23}$$

$$= p_a \dot{q}^a - L \tag{24}$$

This quantity is the Hamiltonian, as defined by a Legendre transformation. The Hamiltonian is usually taken to be the energy E of the system, so we see that Noether's theorem applied to a symmetry under time translation results in energy being the corresponding conserved quantity.

Note that the assumption 18 that the Lagrangian has no explicit time dependence is crucial in this derivation. If L *did* have time dependence, we wouldn't end up with the conserved quantity Q being equal to the energy.

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