TIME-DEPENDENT LAGRANGIAN

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The Lagrangian is usually defined as a function of generalized coordinates q_i and their time derivatives $\dot{q}_i$. However, it sometimes can depend explicitly on the time t, so we have

$$L = L(q_i, \dot{q}_i, t) \tag{1}$$

In this case, its time derivative is

$$\frac{dL}{dt} = \frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial L}{\partial t} \tag{2}$$

We can use the Euler-Lagrange equations

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0 \tag{3}$$

to convert 2 into

$$\begin{aligned}\frac{dL}{dt} &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial L}{\partial t} && (4)\\ &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right) + \frac{\partial L}{\partial t} && (5)\end{aligned}$$

With the canonical momentum defined as

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i} \tag{6}$$

this is

$$\frac{dL}{dt} = \frac{d}{dt}(p_i \dot{q}_i) + \frac{\partial L}{\partial t} \tag{7}$$

or

$$\frac{d}{dt}(p_i \dot{q}_i - L) = -\frac{\partial L}{\partial t} \tag{8}$$

With the usual definition of the Hamiltonian

$$H = p_i \dot{q}_i - L \tag{9}$$

we see that a time-dependent Lagrangian gives the relation

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} \tag{10}$$