TRANSLATION AND ROTATION INVARIANCE

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Reference: Sidney Coleman, *Quantum Field Theory: Lectures of Sidney Coleman* (edited by Bryan Gin-ge Chen *et al.*), World Scientific, 2019. Section 1.2.

Post date: 5 July 2021.

In section 1.2, Coleman considers a relativistic quantum system for a single free, spinless particle. He proposes the time evolution is given by

$$H|\boldsymbol{p}\rangle=\sqrt{|\boldsymbol{p}|^{2}+\mu^{2}}\,|\boldsymbol{p}\rangle \tag{1}$$

where **p** is the particle's 3-momentum and μ is its mass, with H being the proposed relativistic hamiltonian.

He claims that this system is invariant under translation and rotation, and then proceeds to develop the theory of these two invariances.

Translation invariance. For any active translation (that is, a translation in which the system, rather than the operators, is translated) by a 4-vector a^{μ}, there is a unitary operator $U(a)$ that produces the translation. If we denote a state centred at the origin by $|0\rangle$ (note that is *not* the vacuum state, although the same symbol is more commonly used for this in QFT), then the action of $U(a)$ is

$$U(a)|0\rangle=|a\rangle \tag{2}$$

That is, $U(a)$ translates the state so that it is now centred at a. We've seen that the generator of translations is the energy-momentum operator P, so that

$$U(a)=e^{iP\cdot a}=e^{iHt-i\boldsymbol{P}\cdot\boldsymbol{a}} \tag{3}$$

In his eqn 1.37, Coleman defines the translated operator by

$$O(x+a)=U(a)\,O(x)\,U^{\dagger}(a) \tag{4}$$

This appears to be different from the translated operator defined in, for example, Shankar's book, Chapter 11 (Shankar's eqn 11.2.22), where he says that

$$O(x+a)=U^{\dagger}(a)\,O(x)\,U(a) \tag{5}$$

With Coleman's definition 4, we have

$$\langle a|O(x+a)|a\rangle = \left\langle 0\left|U^{\dagger}(a)\,U(a)\,O(x)\,U^{\dagger}(a)\,U(a)\right|0\right\rangle \tag{6}$$

$$= \langle 0|O(x)|0\rangle \tag{7}$$

That is, if we translate both the system and the operator by a, then the expectation value of O at the point a in the translated system is the same as the expectation value of O in the untranslated system. This appears to make sense, since if we transform *everything* by the same amount, nothing should change.

For space translations alone, we take $t=0$ in 3, which gives us

$$U(\boldsymbol{a}) = e^{-i\boldsymbol{P}\cdot\boldsymbol{a}} \tag{8}$$

$$e^{-i\boldsymbol{P}\cdot\boldsymbol{a}}|\boldsymbol{q}\rangle = |\boldsymbol{q}+\boldsymbol{a}\rangle \tag{9}$$

$$e^{-i\boldsymbol{P}\cdot\boldsymbol{a}}O(\boldsymbol{x})\,e^{i\boldsymbol{P}\cdot\boldsymbol{a}} = O(\boldsymbol{x}+\boldsymbol{a}) \tag{10}$$

This translation is stated to apply only for operators O that are localized in space, though I'm not entirely sure what that means.

The position operator transforms differently, however, as Coleman shows in his eqn 1.40, where we find

$$e^{i\boldsymbol{P}\cdot\boldsymbol{a}}\hat{\boldsymbol{q}}e^{-i\boldsymbol{P}\cdot\boldsymbol{a}} = \hat{\boldsymbol{q}}+\boldsymbol{a} \tag{11}$$

In Shankar's book, the position operator and more general operators both transform the same way, as in 5. See Shankar eqns 11.2.5 and 11.2.22.

Rotation invariance. For a rotation R, we again have a unitary operator $U(R)$ that implements the rotation. We define the rotated ket $|\psi'\rangle$ to satisfy

$$\left|\psi'\right\rangle = U(R)\,|\psi\rangle \tag{12}$$

We now consider the effect that a rotation has on the 3-momentum operator $\boldsymbol{P}$. It makes sense that the expectation value of $\boldsymbol{P}$ in the rotated state $|\psi'\rangle$ should be the same as the rotated expectation value of $\boldsymbol{P}$ in the original state. That is

$$\left\langle \psi'|\boldsymbol{P}|\psi'\right\rangle = R\,\langle\psi|\boldsymbol{P}|\psi\rangle \tag{13}$$

Inserting 12 on the LHS, we get

$$\left\langle \psi'|\boldsymbol{P}|\psi'\right\rangle = \left\langle \psi\left|U^{\dagger}(R)\,\boldsymbol{P}U(R)\right|\psi\right\rangle \tag{14}$$

so

$$U^{\dagger}(R)\,\boldsymbol{P}U(R)=R\boldsymbol{P} \tag{15}$$

Here R is just a number (or, rather, a matrix of numbers), so it can be taken inside the bracket.

Coleman shows that if we take

$$U(R)|\boldsymbol{p}\rangle=|R\boldsymbol{p}\rangle \tag{16}$$

then it satisfies the conditions required. That is, we have

$$U(R)\,U^{\dagger}(R)=1 \tag{17}$$

$$U(1)=1 \tag{18}$$

$$U(R_1)\,U(R_2)=U(R_1R_2) \tag{19}$$

The proofs of these are given in Coleman's eqns 1.48 through 1.50.

We can also show that

$$U^{\dagger}(R)\,HU(R)=H \tag{20}$$

where H is the hamiltonian, that is, the component P^0. We have

$$U^{\dagger}(R)\,HU(R)=U^{-1}(R)\,H\left(U^{-1}(R)\right)^{\dagger} \tag{21}$$

$$=U\left(R^{-1}\right)HU^{\dagger}\left(R^{-1}\right) \tag{22}$$

$$=U\left(R^{-1}\right)H\int d^3\boldsymbol{p}\;|\boldsymbol{p}\rangle\langle\boldsymbol{p}|\,U^{\dagger}\left(R^{-1}\right) \tag{23}$$

$$=U\left(R^{-1}\right)\int d^3\boldsymbol{p}\;E_{\boldsymbol{p}}\,|\boldsymbol{p}\rangle\langle\boldsymbol{p}|\,U^{\dagger}\left(R^{-1}\right) \tag{24}$$

$$=\int d^3\boldsymbol{p}\;E_{\boldsymbol{p}}U\left(R^{-1}\right)|\boldsymbol{p}\rangle\langle\boldsymbol{p}|\,U^{\dagger}\left(R^{-1}\right) \tag{25}$$

$$=\int d^3\boldsymbol{p}\;E_{\boldsymbol{p}}\left|R^{-1}\boldsymbol{p}\right\rangle\left\langle R^{-1}\boldsymbol{p}\right| \tag{26}$$

In the third line, we inserted the unit operator in the form

$$1=\int d^3\boldsymbol{p}\;|\boldsymbol{p}\rangle\langle\boldsymbol{p}| \tag{27}$$

In the fourth line, we used the fact that a momentum ket $|\boldsymbol{p}\rangle$ is an eigenstate of the hamiltonian with eigenvalue $E_{\boldsymbol{p}}$, the energy of a particle with momentum $\boldsymbol{p}$:

$$E_{\boldsymbol{p}}=\sqrt{|\boldsymbol{p}|^2+\mu^2} \tag{28}$$

To continue from 26, we can do the substitution $\boldsymbol{p}'=R^{-1}\boldsymbol{p}$ and use the fact that if we rigidly rotate the system, the integration element $d^3\boldsymbol{p}$ doesn't change (or, in more formal language, the Jacobian of the transformation is 1), and also a rigid rotation will not change the energy $E_{\boldsymbol{p}}$. So we have $E_{\boldsymbol{p}'} = E_{\boldsymbol{p}}$ and

$$U^{\dagger}(R) H U(R) = \int d^3\boldsymbol{p}'\, E_{\boldsymbol{p}'} \left|\boldsymbol{p}'\right\rangle \left\langle\boldsymbol{p}'\right| \tag{29}$$

This is just another way of writing the hamiltonian as an expansion over its eigenstates, so the result is that

Note that we are now using Shankar's transformation formula 5 rather than Coleman's original formula 4.

$$U^{\dagger}(R) H U(R) = H \tag{30}$$

and the hamiltonian is invariant under rotation.