TRANSLATION SYMMETRY AND CONSERVATION OF MOMENTUM

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Our first example of applying Noether's theorem is with symmetry under translation.

We consider the Lagrangian

$$L=\frac{1}{2}\sum_{r}m_{r}\dot{\boldsymbol{x}}^{r}\cdot\dot{\boldsymbol{x}}^{r}+\sum_{r>s}V^{(r,s)}\left(\left|\boldsymbol{x}^{r}-\boldsymbol{x}^{s}\right|\right)\tag{1}$$

This describes a system with the usual classical kinetic energy in the first term and a potential energy $V^{(r,s)}$ that depends only on the distances between the particles, and not on their absolute positions. If we consider the transformation

$$\boldsymbol{x}^{r}\to\boldsymbol{x}^{r}+\lambda\boldsymbol{e}\tag{2}$$

where $\boldsymbol{e}$ is a fixed unit vector in some given direction, then we have translated the entire system by a fixed amount in that direction. It's fairly obvious from the Lagrangian 1 that it won't change under this transformation, since $\dot{\boldsymbol{x}}^{r}$ (the derivative of $\boldsymbol{x}^{r}$) won't change, and, since all particles are translated by exactly the same amount, the distances between them won't change either.

To apply Noether's theorem to this symmetry, we need to find

$$DL=\frac{\partial L}{\partial q^{a}}Dq^{a}+\frac{\partial L}{\partial\dot{q}^{a}}D\dot{q}^{a}\tag{3}$$

$$=\frac{\partial L}{\partial q^{a}}Dq^{a}+p_{a}D\dot{q}^{a}\tag{4}$$

where

$$p_{a}\equiv\frac{\partial L}{\partial\dot{q}^{a}}\tag{5}$$

is the momentum conjugate to the coordinate q^{a}, and

$$Dq^a \equiv \left.\frac{\partial q^a}{\partial \lambda}\right|_{\lambda=0} \tag{6}$$

$$D\dot{q}^a = \frac{d}{dt} Dq^a \tag{7}$$

To simplify things, we'll consider the case where we have only 2 particles. The generalization to the multi-particle 3-d case is straightforward, if a bit messy. In this case, the Lagrangian is

$$L = \frac{1}{2}\sum_{r=1}^{2}\left[m_r\left(\dot{x}^r\right)^2 + \left(\dot{y}^r\right)^2 + \left(\dot{z}^r\right)^2\right] + V\left(\sqrt{\left(x^1 - x^2\right)^2 + \left(y^1 - y^2\right)^2 + \left(z^1 - z^2\right)^2}\right) \tag{8}$$

From 2 we have

$$Dx^r = \left.\frac{\partial x^r}{\partial \lambda}\right|_{\lambda=0} \tag{9}$$

$$= e_1 \tag{10}$$

and similarly for the other coordinates. Here, the superscript r indicates particle r, and the subscript 1 indicates the x component the vector $\boldsymbol{e}$.

Since

$$\dot{\boldsymbol{x}}^r \rightarrow \dot{\boldsymbol{x}}^r \tag{11}$$

we have

$$D\dot{x}^1 = \left.\frac{\partial \dot{x}^1}{\partial \lambda}\right|_{\lambda=0} = 0 \tag{12}$$

We also have, after defining

$$r \equiv \sqrt{\left(x^1 - x^2\right)^2 + \left(y^1 - y^2\right)^2 + \left(z^1 - z^2\right)^2} \tag{13}$$

the following, using the chain rule:

$$\frac{\partial L}{\partial x^1} = \frac{\partial V}{\partial r}\frac{\left(x^1 - x^2\right)}{r} \tag{14}$$

$$\frac{\partial L}{\partial x^2} = -\frac{\partial V}{\partial r}\frac{\left(x^1 - x^2\right)}{r} \tag{15}$$

Also

$$\frac{\partial L}{\partial \dot{x}^1} = m_1 \dot{x}^1 \tag{16}$$

and likewise for the other coordinates.

Therefore

$$DL = \frac{\partial L}{\partial q^a} Dq^a + \frac{\partial L}{\partial \dot{q}^a} D\dot{q}^a \tag{17}$$

$$= \frac{\partial V}{\partial r} \frac{\left(x^1 - x^2\right)}{r} e_1 - \frac{\partial V}{\partial r} \frac{\left(x^1 - x^2\right)}{r} e_1 + \ldots + 0 \tag{18}$$

where the $\ldots$ in the second line are the terms involving V for the other coordinates, and the $+0$ is for the term $\frac{\partial L}{\partial \dot{q}^a} D\dot{q}^a$, since $D\dot{q}^a = 0$.

We can see that terms in DL all cancel in pairs, giving the result $DL = 0$ (which is what we'd expect from the physics of the situation as described above). This symmetry results in a conservation law if

$$DL = \frac{dF}{dt} \tag{19}$$

for some function $F\left(q^a, \dot{q}^a, t\right)$.

Since $DL = 0$, then taking $F = 0$ satisfies 19.

The conserved quantity is Q, defined by

$$Q = p_a Dq^a - F \tag{20}$$

so from 20 and 16 we can now find the conserved quantity:

$$Q = p_a Dq^a \tag{21}$$

$$= \left(\frac{\partial L}{\partial \dot{x}^1} + \frac{\partial L}{\partial \dot{x}^2}\right) e_1 + \left(\frac{\partial L}{\partial \dot{y}^1} + \frac{\partial L}{\partial \dot{y}^2}\right) e_2 + \left(\frac{\partial L}{\partial \dot{z}^1} + \frac{\partial L}{\partial \dot{z}^2}\right) e_3 \tag{22}$$

$$= \left(m_1 \dot{x}^1 + m_2 \dot{x}^2\right) e_1 + \left(m_1 \dot{y}^1 + m_2 \dot{y}^2\right) e_2 + \left(m_1 \dot{z}^1 + m_2 \dot{z}^2\right) e_3 \tag{23}$$

$$= m_1 \dot{\boldsymbol{x}}^1 \cdot \boldsymbol{e} + m_2 \dot{\boldsymbol{x}}^2 \cdot \boldsymbol{e} \tag{24}$$

Since the vector $\boldsymbol{e}$ is arbitrary, we get an infinite number of conserved quantities, one for each direction of $\boldsymbol{e}$. However, in 3-d space there are only 3 independent directions (one for each coordinate axis), so we actually get only 3 independent conserved quantities. These quantities can be combined into the single vector

$$\boldsymbol{p} = \sum_r m_r \dot{\boldsymbol{x}}^r \tag{25}$$

which is just the total classical momentum. In other words, for a Lagrangian that is symmetric under translation, the corresponding system has a conserved total momentum.

Coleman's derivation is, of course, much shorter, but it relies on physical intuition rather than explicitly working out the components of DL, so I think it's worth going through the gory details just to see how the system works.

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