WICK'S THEOREM FOR TWO FERMION FIELDS

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Wick's theorem allows us to convert time-ordered products of fields into normal ordered products of fields, together with some contractions. Klauber gives a full example of applying Wick's theorem to a product of two scalar fields in his Chapter 7. Here, we'll run through a similar calculation for a product of two fermion fields.

The general solutions of the Dirac equation are

$$\psi = \sum_{r=1}^{2}\sum_{\mathbf{p}}\sqrt{\frac{m}{VE_{\mathbf{p}}}}\left[c_r(\mathbf{p})\,u_r(\mathbf{p})\,e^{-ipx}+d_r^{\dagger}(\mathbf{p})\,v_r(\mathbf{p})\,e^{ipx}\right] \tag{1}$$

$$\equiv \psi^{+}+\psi^{-} \tag{2}$$

$$\bar{\psi} = \sum_{r=1}^{2}\sum_{\mathbf{p}}\sqrt{\frac{m}{VE_{\mathbf{p}}}}\left[d_r(\mathbf{p})\,\bar{v}_r(\mathbf{p})\,e^{-ipx}+c_r^{\dagger}(\mathbf{p})\,\bar{u}_r(\mathbf{p})\,e^{ipx}\right] \tag{3}$$

$$\equiv \bar{\psi}^{+}+\bar{\psi}^{-} \tag{4}$$

where the coefficients $c_r(\mathbf{p})$ and $d_r(\mathbf{p})$ (and their Hermitian conjugates) are operators rather than numbers. c_r and d_r are destruction operators, and their conjugates $c_r^{\dagger}$ and $d_r^{\dagger}$ are creation operators. It is these operators that obey the anticommutation relations, which as far as I can tell, are just postulates of the theory. The anticommutation relations are

$$\left[c_r(\mathbf{p}),c_s^{\dagger}(\mathbf{p}')\right]_{+}=\left[d_r(\mathbf{p}),d_s^{\dagger}(\mathbf{p}')\right]_{+}=\delta_{rs}\delta_{\mathbf{pp}'} \tag{5}$$

[Note that some books use braces $\{\}$ to indicate anticommutators rather than the $[]_{+}$ notation used by Klauber.] All other anticommutators are taken to be zero.

As a result of this, the only anticommutators that are nonzero are $\left[\psi^{+},\bar{\psi}^{-}\right]_{+}$ and $\left[\psi^{-},\bar{\psi}^{+}\right]_{+}$.

We'll consider the product $\psi(x_1)\bar{\psi}(x_2)$, where the x_i are the 4d space-time coordinates at which the field is evaluated. We wish to find the relation between the time-ordered and normal-ordered forms of this product of 2 operators. Following Klauber, we define T_c to be time ordering, and N_c to be normal ordering, both taking into account the anticommutation relations. To save a bit of space, we'll define

$$\psi_i \equiv \psi(x_i) \tag{6}$$

and similarly for $\bar{\psi}_i$.

Consider first the case $t_2 < t_1$, so that event x_2 occurs before event x_1. In this case we have

$$T_c\left\{\psi_1\bar{\psi}_2\right\} = T_c\left\{\left(\psi_1^+ + \psi_1^-\right)\left(\bar{\psi}_2^+ + \bar{\psi}_2^-\right)\right\} \tag{7}$$

$$= T_c\left\{\psi_1^+\bar{\psi}_2^+ + \psi_1^+\bar{\psi}_2^- + \psi_1^-\bar{\psi}_2^+ + \psi_1^-\bar{\psi}_2^-\right\} \tag{8}$$

Since $t_2 < t_1$ and the T_c operation arranges things so that the earlier time is on the right, this expression is already time-ordered, so we have

$$T_c\left\{\psi_1\bar{\psi}_2\right\} = \psi_1^+\bar{\psi}_2^+ + \psi_1^+\bar{\psi}_2^- + \psi_1^-\bar{\psi}_2^+ + \psi_1^-\bar{\psi}_2^- \tag{9}$$

Now consider normal ordering, which places destruction operators to the right of creation operators. From equations 1 we see that ψ^+ and $\bar{\psi}^+$ contain the destruction operators and ψ^- and $\bar{\psi}^-$ contain the creation operators, so we need to arrange things so that all operators with superscript + are placed to the right of operators with a superscript $-$. We have

$$N_c\left\{\psi_1\bar{\psi}_2\right\} = N_c\left\{\psi_1^+\bar{\psi}_2^+ + \psi_1^+\bar{\psi}_2^- + \psi_1^-\bar{\psi}_2^+ + \psi_1^-\bar{\psi}_2^-\right\} \tag{10}$$

Only the second term on the RHS is out of order. We can fix this by using the anticommutator.

$$\left[\psi_1^+, \bar{\psi}_2^-\right]_+ = \psi_1^+\bar{\psi}_2^- + \bar{\psi}_2^-\psi_1^+ \tag{11}$$

Therefore we have

$$N_c\left\{\psi_1\bar{\psi}_2\right\} = \psi_1^+\bar{\psi}_2^+ - \bar{\psi}_2^-\psi_1^+ + \left[\psi_1^+, \bar{\psi}_2^-\right]_+ + \psi_1^-\bar{\psi}_2^+ + \psi_1^-\bar{\psi}_2^- \tag{12}$$

The two forms 9 and 12 are equivalent ways of expressing the product $\psi_1\bar{\psi}_2$, since we've used the correct anticommutation relations to rearrange the terms. Thus they must be equal.

We now define time ordering $T\{\}$ (without the c subscript) as putting all terms in a product in descending order of time *assuming that all operators anticommute*. Similarly, we define normal ordering $N\{\}$ (without the c subscript) in the same way: we arrange all terms in a product so that all destruction operators appear to the right of all creation operators, *assuming all operators anticommute*. In both cases, swapping two operators introduces a minus sign due to the assumption that they anticommute.

In the case $t_2 < t_1$, applying time ordering didn't change anything, so we have

$$T\left\{\psi_1\bar{\psi}_2\right\} = \psi_1^+\bar{\psi}_2^+ + \psi_1^+\bar{\psi}_2^- + \psi_1^-\bar{\psi}_2^+ + \psi_1^-\bar{\psi}_2^- \tag{13}$$

Normal ordering gives us

$$N\left\{\psi_1\bar{\psi}_2\right\} = \psi_1^+\bar{\psi}_2^+ - \bar{\psi}_2^-\psi_1^+ + \psi_1^-\bar{\psi}_2^+ + \psi_1^-\bar{\psi}_2^- \tag{14}$$

Note that we've swapped the two fields in the second term and introduced a minus sign. Because we're assuming all fields anticommute, there is no anticommutator needed, unlike with 12.

Comparing 12 and 14 we see that

$$N\left\{\psi_1\bar{\psi}_2\right\} = N_c\left\{\psi_1\bar{\psi}_2\right\} + \left[\psi_1^+, \bar{\psi}_2^-\right]_+ \tag{15}$$

Finally, equating T_c and N_c, we have

$$T\left\{\psi_1\bar{\psi}_2\right\} = N\left\{\psi_1\bar{\psi}_2\right\} + \left[\psi_1^+, \bar{\psi}_2^-\right]_+ \tag{16}$$

The last term is an example of a *contraction*, which is just a number due to 5, not an operator. Contractions are defined for fermion fields as

$$\underbracket{\psi_1\bar{\psi}_2} \equiv \begin{cases} \left[\psi_1^+, \bar{\psi}_2^-\right]_+ & t_2 < t_1 \\ -\left[\bar{\psi}_2^+, \psi_1^-\right]_+ & t_1 < t_2 \end{cases} \tag{17}$$

[Some texts use an upper bracket to indicate a contraction. Klauber uses the lower bracket.]

Thus we have, for $t_2 < t_1$

$$T\left\{\psi_1\bar{\psi}_2\right\} = N\left\{\psi_1\bar{\psi}_2\right\} + \underbracket{\psi_1\bar{\psi}_2} \tag{18}$$

For the case $t_1 < t_2$ we have

$$T_c\left\{\psi_1\bar{\psi}_2\right\} = T_c\left\{\psi_1^+\bar{\psi}_2^+ + \psi_1^+\bar{\psi}_2^- + \psi_1^-\bar{\psi}_2^+ + \psi_1^-\bar{\psi}_2^-\right\} \tag{19}$$

We now have to swap every term to get t_1 on the right. The first and last terms anticommute, while the second and third terms can be fixed using anticommutators. We get

$$T_c\left\{\psi_1\bar{\psi}_2\right\} = -\bar{\psi}_2^+\psi_1^+ - \bar{\psi}_2^-\psi_1^+ + \left[\psi_1^+, \bar{\psi}_2^-\right]_+ - \bar{\psi}_2^+\psi_1^- + \left[\psi_1^-, \bar{\psi}_2^+\right]_+ - \bar{\psi}_2^-\psi_1^- \tag{20}$$

The normal ordering gives the same result as with $t_2 < t_1$, since it doesn't depend on the times. Equating 12 and 20 we have

$$\psi_1^+\bar{\psi}_2^+ - \bar{\psi}_2^-\psi_1^+ + \left[\psi_1^+,\bar{\psi}_2^-\right]_+ + \psi_1^-\bar{\psi}_2^+ + \psi_1^-\bar{\psi}_2^- =$$
$$-\bar{\psi}_2^+\psi_1^+ - \bar{\psi}_2^-\psi_1^+ + \left[\psi_1^+,\bar{\psi}_2^-\right]_+ - \bar{\psi}_2^+\psi_1^- + \left[\psi_1^-,\bar{\psi}_2^+\right]_+ - \bar{\psi}_2^-\psi_1^- \quad (21)$$

We can cancel the $\left[\psi_1^+,\bar{\psi}_2^-\right]_+$ from both sides to get

$$\psi_1^+\bar{\psi}_2^+ - \bar{\psi}_2^-\psi_1^+ + \psi_1^-\bar{\psi}_2^+ + \psi_1^-\bar{\psi}_2^- = -\bar{\psi}_2^+\psi_1^+ - \bar{\psi}_2^-\psi_1^+ - \bar{\psi}_2^+\psi_1^- + \left[\psi_1^-,\bar{\psi}_2^+\right]_+ - \bar{\psi}_2^-\psi_1^- \quad (22)$$

The LHS is just $N\left\{\psi_1\bar{\psi}_2\right\}$. If we apply the time ordering T (without the subscript c), we merely swap every term and introduce a minus sign in each case, so we have

$$T\left\{\psi_1\bar{\psi}_2\right\} = -\bar{\psi}_2^+\psi_1^+ - \bar{\psi}_2^-\psi_1^+ - \bar{\psi}_2^+\psi_1^- - \bar{\psi}_2^-\psi_1^- \quad (23)$$

Comparing this with 22 we have

$$N\left\{\psi_1\bar{\psi}_2\right\} = T\left\{\psi_1\bar{\psi}_2\right\} + \left[\psi_1^-,\bar{\psi}_2^+\right]_+ \quad (24)$$

or

$$T\left\{\psi_1\bar{\psi}_2\right\} = N\left\{\psi_1\bar{\psi}_2\right\} - \left[\psi_1^-,\bar{\psi}_2^+\right]_+ \quad (25)$$
$$= N\left\{\psi_1\bar{\psi}_2\right\} - \left[\bar{\psi}_2^+,\psi_1^-\right]_+ \quad (26)$$

where we swapped the order in the anticommutator in the last line, since the order doesn't matter in an anticommutator. Comparing this with 17 we see that, for $t_1 < t_2$

$$T\left\{\psi_1\bar{\psi}_2\right\} = N\left\{\psi_1\bar{\psi}_2\right\} + \underbrace{\psi_1\bar{\psi}_2} \quad (27)$$

That is, 18 is valid for both time orders.

This is an example of Wick's theorem for two fermion fields.

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