

WICK'S THEOREM

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The fundamental problem in the calculation of the S-matrix is to evaluate Dyson's formula

$$U_I(t, t') = T \exp \left(-i \int_{t'}^t dt'' H_I(t'') \right) \quad (1)$$

This is much easier to deal with if we can convert a time-ordered product into normal ordered products. Wick's theorem provides an algorithm for doing just that. In order to describe Wick's theorem, we need to define a few terms.

A *contraction* of two fields is defined as the difference between the time-ordered and normal-ordered version of the fields. A contraction is written by a bracket connecting the two fields. The bracket can be drawn either above or below the fields; Coleman draws it above, so we have

$$\overline{A(x) B(y)} \equiv T(A(x) B(y)) - :A(x) B(y): \quad (2)$$

Coleman shows in eqns 8.15 through 8.20 that a contraction is just a number (or a c-number, as he calls it) and not an operator, so that it commutes with every operator. If we split the fields into their annihilation and creation parts according to

$$A(x) = A^{(+)}(x) + A^{(-)}(x) \quad (3)$$

where $A^{(+)}$ contains the annihilation operators and $A^{(-)}$ the creation operators, then the contraction of two fields $A(x)$ and $B(y)$ is

$$\overline{A(x) B(y)} = [A^{(+)}(x), B^{(-)}(x)] \quad (4)$$

$$= \begin{cases} \Delta_+(x-y) & \text{if } A = B \\ 0 & \text{if } A \neq B \end{cases} \quad (5)$$

where

$$\Delta_+(x-y) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 (2\omega_{\mathbf{p}})} e^{-ip \cdot (x-y)} \quad (6)$$

For a scalar field $\phi(x)$ with particles of mass μ we have, following Coleman's eqns 8.21 to 8.24, and using the result of Coleman's Problem 1.3 (lots of details here, but it's all explained well in the book):

$$\overline{\phi(x)\phi(y)} = \langle 0|T(\phi(x)\phi(y))|0\rangle \quad (7)$$

$$= \lim_{\epsilon \rightarrow 0^+} \int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot (x-y)} \frac{i}{p^2 - \mu^2 + i\epsilon} \quad (8)$$

For the nucleon fields ψ and ψ^* with particles of mass m , we have

$$\overline{\psi^*(x)\psi(y)} = \overline{\psi(x)\psi^*(y)} \quad (9)$$

$$= \lim_{\epsilon \rightarrow 0^+} \int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot (x-y)} \frac{i}{p^2 - m^2 + i\epsilon} \quad (10)$$

with all other contractions being zero.

Coleman then states Wick's theorem. We have

$$T(\phi_1\phi_2 \dots \phi_n) = :\phi_1\phi_2 \dots \phi_n: \quad (11)$$

$$+ \overline{\phi_1\phi_2} \dots \phi_n + (\text{all other terms with 1 contraction}) \quad (12)$$

$$+ \overline{\phi_1\phi_2\phi_3\phi_4} \dots \phi_n + (\text{all other 2 contraction terms}) \quad (13)$$

$$+ \dots + \left(\text{all terms with } \frac{1}{2}n \text{ or } \frac{1}{2}(n-1) \text{ contractions} \right) \quad (14)$$

where the last term uses $\frac{1}{2}n$ if n is even and $\frac{1}{2}(n-1)$ if n is odd.

Coleman gives an inductive proof of Wick's theorem which is fairly clear except for one statement near the end of the proof. For the term

$$\left[\phi_1^{(+)}, W(\phi_2\phi_3 \dots \phi_n) \right] \quad (15)$$

he claims that this contains all possible contractions involving ϕ_1 . Here W is the RHS of Wick's theorem given in 12, which is assumed to be correct for operators ϕ_2, \dots, ϕ_n . This actually does follow from 4 if we expand the commutator and rewrite the terms. I'll give an example so we can see how

it goes. Suppose $n = 3$ so the term in the commutator is $W(\phi_2\phi_3)$. Then Wick's theorem reduces to the first 2 terms in 12:

$$T(\phi_2\phi_3) = :\phi_2\phi_3: + \overline{\phi_2\phi_3} \quad (16)$$

The second term on the RHS is just a number so it commutes with ϕ_1 . Now consider the commutator

$$[\phi_1, \phi_2\phi_3] = \phi_1\phi_2\phi_3 - \phi_2\phi_3\phi_1 \quad (17)$$

$$= [\phi_1, \phi_2]\phi_3 + \phi_2\phi_1\phi_3 - \phi_2[\phi_3, \phi_1] - \phi_2\phi_1\phi_3 \quad (18)$$

$$= [\phi_1, \phi_2]\phi_3 - \phi_2[\phi_3, \phi_1] \quad (19)$$

$$= [\phi_1, \phi_2]\phi_3 + \phi_2[\phi_1, \phi_3] \quad (20)$$

$$= [\phi_1, \phi_2]\phi_3 + [\phi_1, \phi_3]\phi_2 \quad (21)$$

$$= \overline{\phi_1\phi_2\phi_3} + \overline{\phi_1\phi_3\phi_2} \quad (22)$$

In the fourth line, we used the fact that the commutator is a c-number so it commutes with ϕ_2 . Thus we have added a term to $W(\phi_2\phi_3)$ that contains all (2 in this case) contractions involving ϕ_1 . A similar argument would work for products of more than 3 fields, but I won't go into it here.

Finally, Coleman sets an exercise for the reader to show that

$$T(\phi_1\phi_2 \dots \phi_n) = \exp\left(\frac{1}{2} \sum_{i,j=1}^n \overline{\phi_i\phi_j} \frac{\partial}{\partial\phi_i} \frac{\partial}{\partial\phi_j}\right) \phi_1\phi_2 \dots \phi_n \quad (23)$$

To see this, we expand the exponential. The first term in the expansion of an exponential is just 1, so the first term on the RHS of 23 is

$$:\phi_1\phi_2 \dots \phi_n: \quad (24)$$

which matches the first term in Wick's theorem 12.

The next term is

$$\frac{1}{2} \sum_{i,j=1}^n \overline{\phi_i\phi_j} \frac{\partial}{\partial\phi_i} \frac{\partial}{\partial\phi_j} (\phi_1\phi_2 \dots \phi_n) \quad (25)$$

The derivatives remove ϕ_i and ϕ_j from the product $(\phi_1\phi_2 \dots \phi_n)$, so we get the sum of a bunch of terms, each with a single contraction. However, since we're summing over both i and j from 1 to n , we are double counting each term where $i \neq j$. The factor of $\frac{1}{2}$ out front takes care of this. Thus this term reproduces the second term in the Wick expansion 12. (Note that the term in the sum with $i = j$ gives zero, since the contraction of a field with itself is zero.)

We'll do one more term explicitly before considering the general case. The next term in the expansion of the exponential 23 is

$$\frac{1}{2!} \frac{1}{2^2} \sum_{i,j,k,\ell=1}^n \overline{\phi_i \phi_j} \frac{\partial}{\partial \phi_i} \frac{\partial}{\partial \phi_j} \overline{\phi_k \phi_\ell} \frac{\partial}{\partial \phi_k} \frac{\partial}{\partial \phi_\ell} (\phi_1 \phi_2 \dots \phi_n) \quad (26)$$

The 4 derivatives remove ϕ_i , ϕ_j , ϕ_k and ϕ_ℓ from the product of fields. We now have all possible terms with 2 contractions. How much are we overcounting, however? The two subscripts within each contraction can be swapped, so each contraction is counted twice, giving a duplication factor of 2^2 . However, the pair of subscripts in the first contraction can also be swapped with the pair of subscripts in the second contraction, giving another factor of 2, so in total the sum overcounts the number of terms by a factor of $2! \times 2^2 = 8$, which is cancelled by the $\frac{1}{2!} \frac{1}{2^2}$ out front. Thus this term exactly reproduces the term in 12 with 2 contractions.

Now consider the term in the expansion of 23 with m contractions. This term in the expansion will have a numerical factor of $\frac{1}{m!} \frac{1}{2^m}$. There are m pairs of fields, within each of which we can swap the two subscripts, so we overcount by a factor of 2^m . In addition, the m contractions can be reordered in $m!$ ways, so the total overcounting factor is $2^m m!$, which is exactly cancelled by the $\frac{1}{m!} \frac{1}{2^m}$ that arises from the series expansion. Thus this term exactly reproduces the term in Wick's theorem with m contractions.

The series expansion of 23 terminates once we reach either $\frac{n}{2}$ or $\frac{1}{2}(n-1)$ (depending on n being even or odd), since at that point we have more derivatives than terms in the product $(\phi_1 \phi_2 \dots \phi_n)$ so the derivatives will give zero.

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