

## LINEAR CHAIN OF OSCILLATORS - EXTERNAL FORCE, UNITARY OPERATOR

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Reference: W. Greiner & J. Reinhardt, *Field Quantization*, Springer-Verlag (1996), Section 1.5, Example 1.2.

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The Hamiltonian for a quantum system of a linear chain of oscillators is

$$H_0 = \frac{1}{2m} \sum_{n=1}^N p_n^2 + \frac{\kappa}{2} \sum_{n=1}^N (q_{n+1}(t) - q_n(t))^2 \quad (1)$$

In Greiner's Example 1.2, we now consider what happens if we add an external force  $\mathcal{F}_n$ , which is assumed to be constant in time, although it may vary over the length of the chain. The potential energy due to the force  $\mathcal{F}_n$  on oscillator  $n$  is therefore the force times the displacement from equilibrium, the latter of which is  $q_n$ . Thus the potential energy is changed by an amount

$$V_1 = - \sum_n \mathcal{F}_n q_n \quad (2)$$

The minus sign indicates that the force acts to oppose the displacement. The new Hamiltonian is therefore

$$H = H_0 + V_1 \quad (3)$$

If we substitute

$$q_n(t) = \sum_k \sqrt{\frac{\hbar}{2m\omega_k}} \left( c_k u_n^k + c_k^\dagger u_n^{k*} \right) \quad (4)$$

and

$$H_0 = \sum_k \hbar\omega_k \left( c_k^\dagger c_k + \frac{1}{2} \right) \quad (5)$$

then Greiner shows that the new Hamiltonian can be written

$$H = \sum_k \left[ \hbar\omega_k \left( c_k^\dagger c_k + \frac{1}{2} \right) - F_k c_k - F_k^* c_k^\dagger \right] \quad (6)$$

where the  $F_k$  coefficients are purely numerical quantities (not operators):

$$F_k = \sqrt{\frac{\hbar}{2Nm\omega_k}} \sum_n \mathcal{F}_n e^{ikan} \quad (7)$$

The Hamiltonian 6 is not diagonal in the basis of eigenstates of the number operator  $c_k^\dagger c_k$  because of the extra terms. The  $c_k$  is an annihilation operator it deletes an energy quantum from a state on which it operates, and  $c_k^\dagger$  is a creation operator so adds a quantum. Greiner shows that by transforming the operators according to

$$d_k = c_k - \alpha_k \quad (8)$$

$$d_k^\dagger = c_k^\dagger - \alpha_k^* \quad (9)$$

where  $\alpha_k$  is an ordinary c-number (complex number, not an operator), then the Hamiltonian becomes Greiner's eqn (8), which I believe has a typo. It should read

$$H = \sum_k \left( \hbar\omega_k \left( d_k^\dagger d_k + \frac{1}{2} \right) - d_k (F_k - \alpha_k^* \hbar\omega_k) - d_k^\dagger (F_k^* - \alpha_k \hbar\omega_k) + \right. \quad (10)$$

$$\left. \hbar\omega_k |\alpha_k|^2 - \alpha_k F_k - \alpha_k^* F_k^* \right) \quad (11)$$

That is, there should be  $\alpha_k$  (not  $\alpha_k^*$ ) in the last term in the first line.

From here, we can make  $H$  diagonal if we make the last two terms in the first line equal to zero. This works if we choose

$$\alpha_k = \frac{1}{\hbar\omega_k} F_k \quad (12)$$

which results in

$$H = \sum_k \hbar\omega_k \left( d_k^\dagger d_k + \frac{1}{2} \right) - \Delta E \quad (13)$$

with

$$\Delta E \equiv - \sum_k \hbar\omega_k |\alpha_k|^2 \quad (14)$$

Note that  $\Delta E < 0$  no matter what the force is, so the overall energy states are all lowered by this amount.

The operators  $d_k$  and  $d_k^\dagger$  satisfy the same commutation relations as the standard raising and lowering operators, so the energy eigenvalues are

$$E_n = \sum_k \hbar \omega_k \left( n_k + \frac{1}{2} \right) + \Delta E \quad (15)$$

As usual, the annihilation operator deletes the ground state:

$$d_k |0, \alpha\rangle = 0 \quad (16)$$

for all  $k$ . From the normalization properties of the harmonic oscillator raising and lowering operators we can build up higher states using the creation operator in the same way as for the single oscillator treated earlier. We get

$$|n, \alpha\rangle = \prod_k \frac{1}{\sqrt{n_k!}} \left( d_k^\dagger \right)^{n_k} |0, \alpha\rangle \quad (17)$$

Each factor of  $\left( d_k^\dagger \right)^{n_k}$  generates  $n_k$  quanta with frequency  $\omega_k$ , and the  $n$  on the LHS is

$$n = \sum_k n_k \quad (18)$$

Here  $\alpha$  represents the set of all  $\alpha_k$  values. It turns out that the ground state  $|0, \alpha\rangle$  is *not* the same as the ground state  $|0\rangle$  of the unperturbed oscillator; more on this later.

Greiner now shows that we can diagonalize the Hamiltonian in an alternative way by using a unitary transformation  $S(\alpha)$ . He proposes to define  $S$  so that

$$S(\alpha) c_k S^\dagger(\alpha) = c_k + \alpha_k \quad (19)$$

$$S(\alpha) c_k^\dagger S^\dagger(\alpha) = c_k^\dagger + \alpha_k^* \quad (20)$$

As  $S$  is unitary, we must also have

$$S^\dagger(\alpha) = S^{-1}(\alpha) \quad (21)$$

Initially, we just assume that such an operator exists, and Greiner shows that this leads to

$$H' |\Psi'\rangle = E |\Psi'\rangle \quad (22)$$

where

$$H' \equiv S H S^\dagger \quad (23)$$

$$|\Psi'\rangle \equiv S |\Psi\rangle \quad (24)$$

If we convert 6 by left multiplying by  $S$  and right-multiplying by  $S^\dagger$ , and use  $S c_k^\dagger c_k S^\dagger = S c_k^\dagger S^\dagger S c_k S^\dagger$ , we can expand the terms to find that

$$H' = \sum_k \hbar \omega_k \left( c_k^\dagger c_k + \frac{1}{2} \right) - \Delta E \quad (25)$$

Thus  $H'$  has the same form as  $H$  in 13, except that we're still using the original operators  $c_k$  and  $c_k^\dagger$ . Because the two Hamiltonians have the same form, and the operators  $c_k$  and  $c_k^\dagger$  have the same commutation relations as  $d_k$  and  $d_k^\dagger$ , the energy spectrums of the two Hamiltonians are the same, and just as with 17, we can generate higher states using  $c_k^\dagger$ :

$$|n', \alpha\rangle = \prod_k \frac{1}{\sqrt{n_k!}} \left( c_k^\dagger \right)^{n_k} |0', \alpha\rangle \quad (26)$$

where the primes on the  $n$  and  $0$  represent the transformed state, so that

$$|0', \alpha\rangle = S |0, \alpha\rangle \quad (27)$$

and

$$c_k |0', \alpha\rangle = 0 \quad (28)$$

Starting with this last equation, we have

$$c_k |0', \alpha\rangle = c_k S |0, \alpha\rangle \quad (29)$$

Left-multiplying by  $S^\dagger$  and using 19 and 8 we have

$$0 = S^\dagger c_k S |0, \alpha\rangle = S^\dagger \left( S(\alpha) c_k S^\dagger(\alpha) - \alpha_k \right) S |0, \alpha\rangle \quad (30)$$

$$= (c_k - \alpha_k) |0, \alpha\rangle \quad (31)$$

$$= d_k |0, \alpha\rangle \quad (32)$$

We thus regain 16.

In Greiner's eqn (25) he uses similar methods to show that

$$S^\dagger |n', \alpha\rangle = \prod_k \frac{1}{\sqrt{n_k!}} \left( d_k^\dagger \right)^{n_k} |0, \alpha\rangle = |n, \alpha\rangle \quad (33)$$

This shows that the state  $|n', \alpha\rangle$  is the transformed version of the original state  $|n, \alpha\rangle$ :

$$|n', \alpha\rangle = S |n, \alpha\rangle \quad (34)$$

Note that the total number of quanta is the same in both states, so that  $n' = n = \sum_k n_k$ .

We've done all this implicitly assuming that an operator  $S$  can actually be found that satisfies the original conditions 19 and 21. Greiner derives this operator clearly in detail in his eqns (27) to (35), so I don't need to go into the details here. To summarize, we start by representing the unitary operator as an exponential:

$$S = e^\Lambda \tag{35}$$

This leads to the requirement that from 19

$$e^\Lambda c_k e^{-\Lambda} = c_k + \alpha_k \tag{36}$$

We can then expand the exponentials and use the Baker-Campbell-Hausdorff formula to arrive at the final form:

$$S(\alpha) = \exp\left(-\sum_k (\alpha_k c_k^\dagger - \alpha_k^* c_k)\right) \tag{37}$$

Note that the exponent is anti-Hermitian, so that  $\Lambda^\dagger = -\Lambda$ .

PINGBACKS

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