LINEAR CHAIN OF OSCILLATORS - EXTERNAL FORCE, UNITARY OPERATOR

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Reference: W. Greiner & J. Reinhardt, *Field Quantization*, Springer-Verlag (1996), Section 1.5, Example 1.2.

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The Hamiltonian for a quantum system of a linear chain of oscillators is

$$H_0 = \frac{1}{2m}\sum_{n=1}^{N} p_n^2 + \frac{\kappa}{2}\sum_{n=1}^{N}\left(q_{n+1}(t) - q_n(t)\right)^2 \tag{1}$$

In Greiner's Example 1.2, we now consider what happens if we add an external force $\mathcal{F}_n$, which is assumed to be constant in time, although it may vary over the length of the chain. The potential energy due to the force $\mathcal{F}_n$ on oscillator n is therefore the force times the displacement from equilibrium, the latter of which is q_n. Thus the potential energy is changed by an amount

$$V_1 = -\sum_n \mathcal{F}_n q_n \tag{2}$$

The minus sign indicates that the force acts to oppose the displacement. The new Hamiltonian is therefore

$$H = H_0 + V_1 \tag{3}$$

If we substitute

$$q_n(t) = \sum_k \sqrt{\frac{\hbar}{2m\omega_k}}\left(c_k u_n^k + c_k^\dagger u_n^{k*}\right) \tag{4}$$

and

$$H_0 = \sum_k \hbar\omega_k\left(c_k^\dagger c_k + \frac{1}{2}\right) \tag{5}$$

then Greiner shows that the new Hamiltonian can be written

$$H = \sum_k\left[\hbar\omega_k\left(c_k^\dagger c_k + \frac{1}{2}\right) - F_k c_k - F_k^* c_k^\dagger\right] \tag{6}$$

where the F_k coefficients are purely numerical quantities (not operators):

$$F_k = \sqrt{\frac{\hbar}{2Nm\omega_k}} \sum_n \mathcal{F}_n e^{ikan} \tag{7}$$

The Hamiltonian 6 is not diagonal in the basis of eigenstates of the number operator $c_k^\dagger c_k$ because of the extra terms. The c_k is an annihilation operator it deletes an energy quantum from a state on which it operates, and $c^\dagger$ is a creation operator so adds a quantum. Greiner shows that by transforming the operators according to

$$d_k = c_k - \alpha_k \tag{8}$$

$$d_k^\dagger = c_k^\dagger - \alpha_k^* \tag{9}$$

where α_k is an ordinary c-number (complex number, not an operator), then the Hamiltonian becomes Greiner's eqn (8), which I believe has a typo. It should read

$$H = \sum_k \left(\hbar\omega_k \left(d_k^\dagger d_k + \frac{1}{2} \right) - d_k \left(F_k - \alpha_k^* \hbar\omega_k \right) - d_k^\dagger \left(F_k^* - \alpha_k \hbar\omega_k \right) + \right. \tag{10}$$

$$\left. \hbar\omega_k \left|\alpha_k\right|^2 - \alpha_k F_k - \alpha_k^* F_k^* \right) \tag{11}$$

That is, there should be α_k (not α_k^*) in the last term in the first line.

From here, we can make H diagonal if we make the last two terms in the first line equal to zero. This works if we choose

$$\alpha_k = \frac{1}{\hbar\omega_k} F_k \tag{12}$$

which results in

$$H = \sum_k \hbar\omega_k \left(d_k^\dagger d_k + \frac{1}{2} \right) - \Delta E \tag{13}$$

with

$$\Delta E \equiv -\sum_k \hbar\omega_k \left|\alpha_k\right|^2 \tag{14}$$

Note that $\Delta E < 0$ no matter what the force is, so the overall energy states are all lowered by this amount.

The operators d_k and $d_k^\dagger$ satisfy the same commutation relations as the standard raising and lowering operators, so the energy eigenvalues are

$$E_n=\sum_k\hbar\omega_k\left(n_k+\frac{1}{2}\right)+\varDelta E \tag{15}$$

As usual, the annihilation operator deletes the ground state:

$$d_k\left|0,\alpha\right\rangle=0 \tag{16}$$

for all k. From the normalization properties of the harmonic oscillator raising and lowering operators we can build up higher states using the creation operator in the same was a for the single oscillator treated earlier. We get

$$\left|n,\alpha\right\rangle=\prod_k\frac{1}{\sqrt{n_k!}}\left(d_k^{\dagger}\right)^{n_k}\left|0,\alpha\right\rangle \tag{17}$$

Each factor of $\left(d_k^{\dagger}\right)^{n_k}$ generates n_k quanta with frequency ω_k, and the n on the LHS is

$$n=\sum_k n_k \tag{18}$$

Here α represents the set of all α_k values. It turns out that the ground state $\left|0,\alpha\right\rangle$ is *not* the same as the ground state $\left|0\right\rangle$ of the unperturbed oscillator; more on this later.

Greiner now shows that we can diagonalize the Hamiltonian in an alternative way by using a unitary transformation $S(\alpha)$. He proposes to define S so that

$$S(\alpha)\,c_kS^{\dagger}(\alpha)=c_k+\alpha_k \tag{19}$$

$$S(\alpha)\,c_k^{\dagger}S^{\dagger}(\alpha)=c_k^{\dagger}+\alpha_k^* \tag{20}$$

As S is unitary, we must also have

$$S^{\dagger}(\alpha)=S^{-1}(\alpha) \tag{21}$$

Initially, we just assume that such an operator exists, and Greiner shows that this leads to

$$H'\left|\Psi'\right\rangle=E\left|\Psi'\right\rangle \tag{22}$$

where

$$H'\equiv SHS^{\dagger} \tag{23}$$

$$\left|\Psi'\right\rangle\equiv S\left|\Psi\right\rangle \tag{24}$$

If we convert 6 by left multiplying by S and right-multiplying by $S^{\dagger}$, and use $Sc_k^{\dagger}c_kS^{\dagger} = Sc_k^{\dagger}S^{\dagger}Sc_kS^{\dagger}$, we can expand the terms to find that

$$H' = \sum_k \hbar\omega_k \left(c_k^{\dagger}c_k + \frac{1}{2}\right) - \Delta E \tag{25}$$

Thus H' has the same form as H in 13, except that we're still using the original operators c_k and $c_k^{\dagger}$. Because the two Hamiltonians have the same form, and the operators c_k and $c_k^{\dagger}$ have the same commutation relations as d_k and $d_k^{\dagger}$, the energy spectrums of the two Hamiltonians are the same, and just as with 17, we can generate higher states using $c_k^{\dagger}$:

$$\left|n', \alpha\right\rangle = \prod_k \frac{1}{\sqrt{n_k!}} \left(c_k^{\dagger}\right)^{n_k} \left|0', \alpha\right\rangle \tag{26}$$

where the primes on the n and 0 represent the transformed state, so that

$$\left|0', \alpha\right\rangle = S\left|0, \alpha\right\rangle \tag{27}$$

and

$$c_k\left|0', \alpha\right\rangle = 0 \tag{28}$$

Starting with this last equation, we have

$$c_k\left|0', \alpha\right\rangle = c_kS\left|0, \alpha\right\rangle \tag{29}$$

Left-multiplying by $S^{\dagger}$ and using 19 and 8 we have

$$0 = S^{\dagger}c_kS\left|0, \alpha\right\rangle = S^{\dagger}\left(S(\alpha)c_kS^{\dagger}(\alpha) - \alpha_k\right)S\left|0, \alpha\right\rangle \tag{30}$$

$$= (c_k - \alpha_k)\left|0, \alpha\right\rangle \tag{31}$$

$$= d_k\left|0, \alpha\right\rangle \tag{32}$$

We thus regain 16.

In Greiner's eqn (25) he uses similar methods to show that

$$S^{\dagger}\left|n', \alpha\right\rangle = \prod_k \frac{1}{\sqrt{n_k!}} \left(d_k^{\dagger}\right)^{n_k} \left|0, \alpha\right\rangle = \left|n, \alpha\right\rangle \tag{33}$$

This shows that the state $|n', \alpha\rangle$ is the transformed version of the original state $|n, \alpha\rangle$:

$$\left|n', \alpha\right\rangle = S\left|n, \alpha\right\rangle \tag{34}$$

Note that the total number of quanta is the same in both states, so that $n' = n = \sum_k n_k$.

We've done all this implicitly assuming that an operator S can actually be found that satisfies the original conditions 19 and 21. Greiner derives this operator clearly in detail in his eqns (27) to (35), so I don't need to go into the details here. To summarize, we start by representing the unitary operator as an exponential:

$$S = e^{\Lambda} \tag{35}$$

This leads to the requirement that from 19

$$e^{\Lambda} c_k e^{-\Lambda} = c_k + \alpha_k \tag{36}$$

We can then expand the exponentials and use the Baker-Campbell-Hausdorff formula to arrive at the final form:

$$S(\alpha) = \exp\left(-\sum_k \left(\alpha_k c_k^{\dagger} - \alpha_k^* c_k\right)\right) \tag{37}$$

Note that the exponent is anti-Hermitian, so that $\Lambda^{\dagger} = -\Lambda$.

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