LINEAR CHAIN OF OSCILLATORS - EXTERNAL FORCE, GROUND STATE

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Reference: W. Greiner & J. Reinhardt, *Field Quantization*, Springer-Verlag (1996), Section 1.5, Example 1.2.

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In the last post, we saw that the Hamiltonian for a chain of oscillators perturbed by an external force is

$$H_0 = \sum_k \left[\hbar\omega_k \left(c_k^\dagger c_k + \frac{1}{2}\right) - F_k c_k - F_k^* c_k^\dagger\right] \tag{1}$$

where the F_k coefficients are purely numerical quantities (not operators):

$$F_k = \sqrt{\frac{\hbar}{2Nm\omega_k}} \sum_n \mathcal{F}_n e^{ikan} \tag{2}$$

and $\mathcal{F}_n$ is the force on oscillator n. The catch is that in order to diagonalize the perturbed Hamiltonian 1, we had to introduce a couple of new annihilation and creation operators defined by

$$d_k = c_k - \alpha_k \tag{3}$$

$$d_k^\dagger = c_k^\dagger - \alpha_k^* \tag{4}$$

where the α_ks are c-numbers. Using these operators, the Hamiltonian becomes

$$H = \sum_k \hbar\omega_k \left(d_k^\dagger d_k + \frac{1}{2}\right) - \Delta E \tag{5}$$

where ΔE is an ordinary number (not an operator) defined by

$$\Delta E \equiv -\sum_k \hbar\omega_k \left|\alpha_k\right|^2 \tag{6}$$

The ground state $|0,\alpha\rangle$ is defined by the condition that it is annihilated by the operator d_k:

$$d_k |0,\alpha\rangle = 0 \tag{7}$$

As we've seen with the single oscillator, this condition allows us to construct the ground state wave function as a function of the position of the oscillator. However, because $d_k \neq c_k$, the ground states of the perturbed and unperturbed systems will not be the same, as we'd probably expect. We want to find $|0,\alpha\rangle$ in terms of the eigenstates of the unperturbed system.

We also saw in the previous post, that if we use a unitary operator S to transform the system according to

$$H' \equiv SHS^{\dagger} \tag{8}$$

$$\left|\Psi'\right\rangle \equiv S\left|\Psi\right\rangle \tag{9}$$

then the transformed Hamiltonian H' can be written as

$$H' = \sum_k \hbar\omega_k \left(c_k^{\dagger}c_k + \frac{1}{2}\right) - \Delta E \tag{10}$$

That is, H' has the same form as 5 except that it contains the original annihilation and creation operators c_k and $c_k^{\dagger}$. The ground state $|0',\alpha\rangle$ of the transformed system is defined by the condition

$$c_k\left|0',\alpha\right\rangle = 0 \tag{11}$$

We won't need this explicit form here, but it's important to note that the condition 11 is what is used to derive this function, so any state that obeys this equation must be the ground state of the perturbed system. The key point here is that the operators c_k and $c_k^{\dagger}$ are the same operators used in the description of the unperturbed system, and the definition of the ground state $|0\rangle$ of the unperturbed system is also given by

$$c_k\left|0\right\rangle = 0 \tag{12}$$

That is, the ground states of the perturbed and unperturbed systems are both defined by the same condition, so they must be equal:

$$\left|0',\alpha\right\rangle = \left|0\right\rangle \tag{13}$$

Of course, the transformed Hamiltonian H' isn't the same as the actual perturbed Hamiltonian, so we don't really want the eigenstates of H'. However, we can use $|0',\alpha\rangle$ as a link between the unperturbed and original perturbed systems.

Greiner goes through the details in his Example 1.2, eqns (37) to (44). The derivation relies on the Baker-Campbell-Hausdorff relation to work out the transformed ground state. The explicit form of the unitary operator S is

$$S(\alpha) = \exp\left(-\sum_k \left(\alpha_k c_k^\dagger - \alpha_k^* c_k\right)\right) \tag{14}$$

so applying it to the unperturbed ground state, we have

$$S|0\rangle = \exp\left(-\sum_k \left(\alpha_k c_k^\dagger - \alpha_k^* c_k\right)\right)|0\rangle \tag{15}$$

As a result, we can also factor the perturbed ground state:

$$|0,\alpha\rangle = \prod_k |0,\alpha_k\rangle \tag{16}$$

We can then use the BCH formula to isolate the factor that contains only c_ks, since from 12, this will give 0 when acting on $|0\rangle$. Also, because any c_k or $c_k^\dagger$ commutes with any $c_{k'}$ or $c_{k'}^\dagger$ if $k \neq k'$, we can use BCH to factor $S(\alpha)$ into a product of exponentials:

$$S(\alpha) = \prod_k \exp\left(-\left(\alpha_k c_k^\dagger - \alpha_k^* c_k\right)\right) \equiv \prod_k S(\alpha_k) \tag{17}$$

Applying this to $|0\rangle$ gives (see Greiner eqns (41-44) for details):

$$|0,\alpha_k\rangle = S(\alpha_k)|0\rangle \tag{18}$$

$$= e^{-\frac{1}{2}|\alpha_k|^2}\sum_{n_k}\frac{1}{\sqrt{n_k!}}\alpha_k^{n_k}|n_k\rangle \tag{19}$$

where $|n_k\rangle$ is the unperturbed state containing n_k energy quanta (or phonons, as Greiner calls them) of frequency ω_k. Note that $|0,\alpha_k\rangle$ is *not* the complete perturbed ground state; rather it is the factor of that ground state corresponding to phonons of frequency ω_k. This factor is thus a superposition of unperturbed states with various numbers of phonons of that frequency. What's interesting is that the ground state of the perturbed system contains excited states of the unperturbed system. Viewed mathematically, we're just expressing one function $|0,\alpha_k\rangle$ in terms of a linear combination of functions $|n_k\rangle$ from a different basis, so in that sense, it's not particularly surprising.

In Greiner's eqn (45), it is shown that the probability of finding the state $|n_k\rangle$ in the perturbed ground state factor $|0,\alpha_k\rangle$ is

$$|\langle n_k|0,\alpha_k\rangle|^2 = e^{-|\alpha_k|^2}\frac{|\alpha_k|^{2n_k}}{n_k!} \tag{20}$$

This is a Poisson distribution, which is more usually found when finding the probability of finding n_k events in a specified time interval, where α_k is the average number of events in that time interval.

The mean number of phonons of type k in $|0,\alpha_k\rangle$ is found from the expectation value of the number operator (see Greiner's eqn (46)):

$$\langle n_k\rangle = \left\langle 0,\alpha_k \left| c_k^{\dagger} c_k \right| 0,\alpha_k \right\rangle = |\alpha_k|^2 \tag{21}$$

To relate this to the perturbing force, we recall the definition of α_k.

$$\alpha_k = \frac{1}{\hbar\omega_k} F_k \tag{22}$$

$$= \frac{1}{\hbar\omega_k}\sqrt{\frac{\hbar}{2Nm\omega_k}}\sum_n \mathcal{F}_n e^{ikan} \tag{23}$$

α_k is effectively the coefficient of frequency ω_k in a discrete Fourier transform, so 21 says that the average number of phonons of type k increases as the Fourier component for frequency ω_k increases, which seems to make sense.