FUNCTIONAL DERIVATIVES AND THE LAGRANGIAN

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Reference: W. Greiner & J. Reinhardt, *Field Quantization*, Springer-Verlag (1996), Chapter 2.

Although I've looked at functionals and functional derivatives before, I've been reading yet another book on quantum field theory (Greiner & Reinhardt, referenced above) so I think it's worth re-examining them and their use in deriving the Euler-Lagrange equations in classical field theory.

G & R's starting point is their definition of a functional derivative (in one dimension) as follows:

$$\delta F[\phi]\equiv\int dx\frac{\delta F[\phi]}{\delta\phi(x)}\delta\phi(x) \tag{1}$$

I find this notation a bit confusing, but I think we can interpret it like this. $F[\phi]$ is a functional of the function $\phi(x)$, which means that F depends on the function ϕ, but not (at least explicitly) on x. One way this might happen is if F is defined as an integral of ϕ over some range of x:

$$F[\phi]=\int_{x_1}^{x_2}\phi(x)\,dx \tag{2}$$

Clearly if we change ϕ, the value of $F[\phi]$ will (usually) change as well, so we can see that a functional is a mapping from a set of *functions* onto the set of ordinary (possibly real or complex) numbers. In this way, it differs from a regular function $f(x)$ which is a mapping from one set of numbers (the x values) to another (usually the same) set of numbers.

To understand what the definition 1 is saying, we need to picture what happens if we perturb the function $\phi(x)$ by an amount $\delta\phi(x)$. At each point x, the perturbation $\delta\phi(x)$ will cause a change in the resulting functional F. This change can be written as $\frac{\delta F[\phi]}{\delta\phi(x)}$, which can be interpreted as the change in F per unit change in ϕ and per unit of x. Since the actual change in ϕ at point x is $\delta\phi(x)$, then the change in $F[x]$ over a distance dx is $dx\frac{\delta F[\phi]}{\delta\phi(x)}\delta\phi(x)$ and the total change in $F[x]$ is the integral over all x, as given by 1.

The reason the notation is confusing is that the functional derivative $\frac{\delta F[\phi]}{\delta\phi(x)}$ makes no mention of the 'per unit of x' part, and as a result, the units in 1 don't appear to balance on each side of the equation. Once we include this, we see that the units of $\frac{\delta F[\phi]}{\delta\phi(x)}$ are $(\text{units of }F)(\text{units of }\phi)^{-1}(\text{length})^{-1}$ and then the units on the RHS of 1 come out to just the units of F, thus agreeing with the LHS.

Another point worth making is that, because the functional derivative *does* depend explicitly on x, it's just an ordinary function, not a functional.

Once we understand this, the derivation of the Euler-Lagrange equations given in G & R's section 2.1 is somewhat easier to follow. Although I've run through this derivation previously, it's worth following through G & R's derivation, as it's a bit different and illustrates the points above.

In classical field theory, the Lagrangian is taken to be a functional of a field function $\phi(\mathbf{x},t)$ and its time derivative $\dot{\phi}(\mathbf{x},t)$:

$$L(t)=L\left[\phi(\mathbf{x},t),\dot{\phi}(\mathbf{x},t)\right] \tag{3}$$

Note a couple of things: first, we're now dealing with 3 space dimensions and second, L has no explicit dependence on the spatial position $\mathbf{x}$. We can generalize 1 to the 3-d case, where L depends on two fields (ϕ and $\dot{\phi}$) by writing

$$\delta L\left[\phi,\dot{\phi}\right]=\int d^3x\left[\frac{\delta L}{\delta\phi(\mathbf{x},t)}\delta\phi(\mathbf{x},t)+\frac{\delta L}{\delta\dot{\phi}(\mathbf{x},t)}\delta\dot{\phi}(\mathbf{x},t)\right] \tag{4}$$

The two functional derivatives on the RHS have the units of $(\text{energy})(\text{volume})^{-1}(\text{units of }\phi)^{-1}$, which again isn't entirely obvious from the notation. At this point, I think it makes a bit more sense to acknowledge the fact that the functional derivatives have the dimensions of a density (that is, the units of something per unit volume), and introduce now the Lagrangian density $\mathcal{L}$, which is the Lagrangian per unit volume, so that the total Lagrangian is defined as

$$L(t)=\int d^3x\,\mathcal{L}\left(\phi(\mathbf{x},t),\nabla\phi(\mathbf{x},t),\dot{\phi}(\mathbf{x},t)\right) \tag{5}$$

Note that $\mathcal{L}$ is actually a function rather than a functional, in that it depends explicitly on the position vector $\mathbf{x}$ via the field function ϕ. That is, the total Lagrangian L does *not* depend on $\mathbf{x}$, since it is an integral over all space, while the Lagrangian density $\mathcal{L}$ is a local function that varies as we move around in space. The list of arguments of $\mathcal{L}\left(\phi(\mathbf{x},t),\nabla\phi(\mathbf{x},t),\dot{\phi}(\mathbf{x},t)\right)$ is an assumption of the theory; we're explicitly assuming that $\mathcal{L}$ doesn't depend on derivatives of any higher order than the first derivative.

We can now write the variation in L using ordinary derivatives:

$$\delta L(t)=\int d^3x\,\left[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi+\frac{\partial\mathcal{L}}{\partial\phi_{,i}}\delta\phi_{,i}+\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\delta\dot{\phi}\right] \tag{6}$$

where the notation $\phi_{,i}$ is defined as

$$\phi_{,i} \equiv \frac{\partial \phi}{\partial x^i} \tag{7}$$

and $i = 1, 2, 3$ with the summation convention being used. We can integrate the middle term in 6 by parts, using

$$\delta\phi_{,i} = \frac{\partial}{\partial x^i}(\delta\phi) \tag{8}$$

$$\int d^3x \frac{\partial \mathcal{L}}{\partial \phi_{,i}} \delta\phi_{,i} = \frac{\partial \mathcal{L}}{\partial \phi_{,i}} \delta\phi \bigg|_{boundary} - \int d^3x \frac{\partial}{\partial x^i}\left(\frac{\partial \mathcal{L}}{\partial \phi_{,i}}\right)\delta\phi \tag{9}$$

As usual, we assume the boundary term goes to zero at infinity, so we're left with

$$\delta L(t) = \int d^3x \left[\left(\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x^i}\left(\frac{\partial \mathcal{L}}{\partial \phi_{,i}}\right)\right)\delta\phi + \frac{\partial \mathcal{L}}{\partial \dot{\phi}}\delta\dot{\phi}\right] \tag{10}$$

Comparing with 4, this allows us to write explicit expressions for the functional derivatives in 4, which makes the volume dependence a bit more obvious:

$$\frac{\delta L}{\delta \phi(\mathbf{x}, t)} = \frac{\partial \mathcal{L}}{\partial \phi(\mathbf{x}, t)} - \frac{\partial}{\partial x^i}\left(\frac{\partial \mathcal{L}}{\partial \phi(\mathbf{x}, t)_{,i}}\right) \tag{11}$$

$$\frac{\delta L}{\delta \dot{\phi}(\mathbf{x}, t)} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(\mathbf{x}, t)} \tag{12}$$

From here, the derivation of the Euler-Lagrange equation proceeds as we showed earlier with the result

$$\frac{\partial \mathcal{L}}{\partial \phi^r} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial \phi^r_{,\mu}}\right) = 0 \tag{13}$$

Here the superscript r labels the field (if we have more than one independent field) and we've used four-vector notation $x^\mu = (t, \mathbf{x})$, and the index μ now extends over 0, 1, 2, 3, with $x^0 = t$.

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