HAMILTON'S EQUATIONS OF MOTION IN CLASSICAL FIELD THEORY

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Reference: W. Greiner & J. Reinhardt, *Field Quantization*, Springer-Verlag (1996), Chapter 2, Section 2.2.

We've looked at the derivation of Hamilton's equations of motion for fields before, but it's worth running through G & R's discussion, and filling in a few gaps.

We start with the Lagrangian as a functional of the field ϕ and its time derivative $\dot{\phi}$:

$$L(t)=L\left[\phi(\mathbf{x},t),\dot{\phi}(\mathbf{x},t)\right] \tag{1}$$

Recall that L does not depend on the spatial position $\mathbf{x}$ as it is actually an integral of the Lagrange density $\mathcal{L}$ (which is an ordinary function, not a functional) over space:

$$L(t)=\int d^3x\,\mathcal{L}\left(\phi(\mathbf{x},t),\nabla\phi(\mathbf{x},t),\dot{\phi}(\mathbf{x},t)\right) \tag{2}$$

To get Hamilton's formulation in classical field theory, we want an analogue of Hamilton's equations in classical particle theory, where we define a conjugate momentum p_k for each position coordinate q_k as

$$p_k\equiv\frac{\partial L}{\partial\dot{q}_k} \tag{3}$$

[Remember that in particle theory, L is an ordinary function, not a functional.]

We then define the Hamiltonian by means of the Legendre transformation

$$H\equiv\sum_k p_k\dot{q}_k-L \tag{4}$$

To do the same thing for fields, we replace the position coordinates q_k by the field functions $\phi(\mathbf{x},t)$, but we then need an analogue for the conjugate momenta p_k. Using 3 as inspiration, and remembering that L in field theory is a functional, not a function, so we need to use a functional derivative instead of the partial derivative in 3. We get the conjugate momentum $\pi(\mathbf{x},t)$:

$$\pi(\mathbf{x},t) \equiv \frac{\delta L}{\delta\dot{\phi}(\mathbf{x},t)} \tag{5}$$

Note that both $\pi(\mathbf{x},t)$ and $\phi(\mathbf{x},t)$ are functions of space and time. Remember also that functional derivatives are actually densities, in that they are given per unit volume.

Now we can define a Legendre transformation for field theory as an analogue of 4 to get the Hamiltonian H, which is a functional:

$$\begin{aligned} H(t) &\equiv \int d^3x\, \pi(\mathbf{x},t)\,\dot{\phi}(\mathbf{x},t) - L(t) && (6) \\ &= \int d^3x \left[\pi(x)\,\dot{\phi}(x) - \mathcal{L}(x)\right] && (7) \end{aligned}$$

where in the last line we used the Lagrangian density 2, and used the single symbol x to represent both space **x** and time t. In classical particle theory, the Hamiltonian is defined to depend on p_k and q_k as the independent variables, rather than q_k and $\dot{q}_k$ on which the Lagrangian depends. For simple particle systems where $p_k = m\dot{q}_k$ (with m being the mass of the particle), this change of variable is fairly trivial, although I imagine there are more complex cases where the correspondence isn't quite so simple.

We can now derive Hamilton's equations of motion by taking the variation of H and using the definition of a functional derivative, and taking the field Hamiltonian to be a functional depending on π and ϕ:

$$\delta H = \int d^3x \left[\frac{\delta H}{\delta\pi}\delta\pi + \frac{\delta H}{\delta\phi}\delta\phi\right] \tag{8}$$

We can find expressions for the two functional derivatives in this equation by finding the variation of 6.

$$\delta H = \int d^3x \left(\dot{\phi}\delta\pi + \pi\delta\dot{\phi}\right) - \delta L \tag{9}$$

The variation of the Lagrangian is

$$\delta L = \int d^3x \left[\frac{\delta L}{\delta\dot{\phi}}\delta\dot{\phi} + \frac{\delta L}{\delta\phi}\delta\phi\right] \tag{10}$$

We can get rid of the two functional derivatives as follows. For the first one, we use 5, while for the second one we can use the Euler-Lagrange equation

$$\frac{\delta L}{\delta\phi}-\frac{\partial}{\partial t}\frac{\delta L}{\delta\dot{\phi}} = 0 \tag{11}$$
$$\frac{\delta L}{\delta\phi} = \dot{\pi} \tag{12}$$

Therefore we have

$$\delta L=\int d^3x\left(\pi\delta\dot{\phi}+\dot{\pi}\delta\phi\right) \tag{13}$$

Inserting this into 9 gives

$$\delta H=\int d^3x\left(\dot{\phi}\delta\pi-\dot{\pi}\delta\phi\right) \tag{14}$$

Comparing with 8 we get

$$\dot{\phi} = \frac{\delta H}{\delta\pi} \tag{15}$$
$$\dot{\pi} = -\frac{\delta H}{\delta\phi} \tag{16}$$

These are Hamilton's equations of motion, although not in a terribly useful form, since we need to know the functional derivatives in order to get a pair of differential equations that we can try to solve. To get an expression in terms of ordinary functions, we can define the Hamiltonian density $\mathcal{H}$ from 7:

$$\mathcal{H}\equiv\pi(x)\,\dot{\phi}(x)-\mathcal{L}(x) \tag{17}$$

As with the Lagrangian density, we now assume that $\mathcal{H}$ depends on π, ϕ and their first spatial derivatives $\partial_i\pi$ and $\partial_i\phi$. We can then follow the same procedure we used earlier to get expressions for the functional derivatives of the Lagrangian in terms of ordinary derivatives of the Lagrangian density. We write the variation of the Hamiltonian as

$$\delta H=\int d^3x\left[\frac{\partial\mathcal{H}}{\partial\pi}\delta\pi+\frac{\partial\mathcal{H}}{\partial\pi_{,i}}\delta\pi_{,i}+\frac{\partial\mathcal{H}}{\partial\phi}\delta\phi+\frac{\partial\mathcal{H}}{\partial\phi_{,i}}\delta\phi_{,i}\right] \tag{18}$$

The second and fourth terms in the integrand can be converted using integration by parts to give (assuming that $\mathcal{H}$ and its derivatives go to zero sufficiently quickly at infinity):

$$\int d^3x\,\frac{\partial\mathcal{H}}{\partial\pi_{,i}}\delta\pi_{,i}=-\int d^3x\frac{\partial}{\partial x^i}\left(\frac{\partial\mathcal{H}}{\partial\pi_{,i}}\right)\delta\pi \tag{19}$$

$$\int d^3x\,\frac{\partial\mathcal{H}}{\partial\phi_{,i}}\delta\phi_{,i}=-\int d^3x\frac{\partial}{\partial x^i}\left(\frac{\partial\mathcal{H}}{\partial\phi_{,i}}\right)\delta\phi \tag{20}$$

We therefore have

$$\delta H=\int d^3x\left[\left(\frac{\partial\mathcal{H}}{\partial\pi}-\frac{\partial}{\partial x^i}\left(\frac{\partial\mathcal{H}}{\partial\pi_{,i}}\right)\right)\delta\pi+\left(\frac{\partial\mathcal{H}}{\partial\phi}-\frac{\partial}{\partial x^i}\left(\frac{\partial\mathcal{H}}{\partial\phi_{,i}}\right)\right)\delta\phi\right] \tag{21}$$

Comparing with 8 we have

$$\frac{\delta H}{\delta\pi} = \frac{\partial\mathcal{H}}{\partial\pi}-\frac{\partial}{\partial x^i}\left(\frac{\partial\mathcal{H}}{\partial\pi_{,i}}\right) \tag{22}$$

$$\frac{\delta H}{\delta\phi} = \frac{\partial\mathcal{H}}{\partial\phi}-\frac{\partial}{\partial x^i}\left(\frac{\partial\mathcal{H}}{\partial\phi_{,i}}\right) \tag{23}$$

Inserting into 15 and 16 we get Hamilton's equations of motion in a form free of functional derivatives:

$$\dot{\phi} = \frac{\partial\mathcal{H}}{\partial\pi}-\frac{\partial}{\partial x^i}\left(\frac{\partial\mathcal{H}}{\partial\pi_{,i}}\right) \tag{24}$$

$$\dot{\pi} = -\frac{\partial\mathcal{H}}{\partial\phi}+\frac{\partial}{\partial x^i}\left(\frac{\partial\mathcal{H}}{\partial\phi_{,i}}\right) \tag{25}$$

These results differ from the ones we got earlier in that the equation for $\dot{\phi}$ has an extra term $-\frac{\partial}{\partial x^i}\left(\frac{\partial\mathcal{H}}{\partial\pi_{,i}}\right)$ that wasn't present earlier. This is because in our earlier treatment, we assumed that $\mathcal{H}$ depended only on ϕ, $\phi_{,i}$ and π but not on $\pi_{,i}$.

If we can get an expression for $\mathcal{H}$ in terms of the field and momentum, and their spatial derivatives, we have a set of differential equations that we can solve (or at least try to).

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