POISSON BRACKETS AND HAMILTON'S EQUATIONS OF MOTION

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Reference: W. Greiner & J. Reinhardt, *Field Quantization*, Springer-Verlag (1996), Chapter 2, Section 2.2.

Although I've looked at Poisson brackets before, it's worth going through G & R's treatment as it is a fair bit simpler and gives clearer results.

First, we need the time derivative of a functional. In the simplest case, a functional $F[\phi]$ depends on a function ϕwhich in turn depends on an independent variable x. The functional itself does not depend on x, however, usually because F is defined as the integral of $\phi(x)$ over some range of x values, so the dependence on x disappears in the integration.

We can generalize things a bit by taking ϕ as a function of two variables, say x and t. If F is defined in the same way (say, as an integral of ϕ over x), then the variable t also appears in the functional, so we can write this as $F(t)$, which is

$$F(t)=\int dx\,g(\phi(x,t)) \tag{1}$$

where $g(\phi)$ is some function of ϕ. Since F now depends on t, we can take the derivative dF/dt which comes out to

$$\dot{F}\equiv\frac{dF}{dt}=\int dx\frac{dg}{d\phi}\frac{\partial\phi}{\partial t}=\int dx\frac{dg}{d\phi}\dot{\phi}(x,t) \tag{2}$$

As we've seen before, the functional derivative of F in this case is

$$\frac{\delta F(t)}{\delta\phi(y,t)}=\frac{dg(\phi(y,t))}{d\phi} \tag{3}$$

where the notation means that we evaluate the derivative on the RHS at the point (y,t). Using this result, we can therefore write $\dot{F}$ as

$$\dot{F}(t)=\int dx\frac{\delta F(t)}{\delta\phi(x,t)}\dot{\phi}(x,t) \tag{4}$$

We can generalize this to 4-d space time, so that x now indicates the four-vector $x=(\mathbf{x},t)$, and the integral is over 3-d space:

$$\dot{F}(t)=\int d^3\mathbf{x}\frac{\delta F(t)}{\delta\phi(x)}\dot{\phi}(x) \tag{5}$$

Generalizing even further, we can make F a functional of two fields, ϕ and π, so we get

$$\dot{F}(t)=\int d^3\mathbf{x}\left[\frac{\delta F(t)}{\delta\phi(x)}\dot{\phi}(x)+\frac{\delta F(t)}{\delta\pi(x)}\dot{\pi}(x)\right] \tag{6}$$

Interpreting ϕ as the field and π as its conjugate momentum, we can now use Hamilton's equations of motion

$$\dot{\phi} = \frac{\delta H}{\delta\pi} \tag{7}$$

$$\dot{\pi} = -\frac{\delta H}{\delta\phi} \tag{8}$$

We get

$$\dot{F}(t)=\int d^3\mathbf{x}\left[\frac{\delta F(t)}{\delta\phi(x)}\frac{\delta H}{\delta\pi}-\frac{\delta F(t)}{\delta\pi(x)}\frac{\delta H}{\delta\phi}\right] \tag{9}$$

The quantity on the RHS is defined to be the Poisson bracket:

$$\{F,H\}_{PB}\equiv\int d^3\mathbf{x}\left[\frac{\delta F(t)}{\delta\phi(x)}\frac{\delta H}{\delta\pi}-\frac{\delta F(t)}{\delta\pi(x)}\frac{\delta H}{\delta\phi}\right] \tag{10}$$

We thus have the general result that the time derivative of a functional is equal to its Poisson bracket with the Hamiltonian:

$$\boxed{\dot{F}(t)=\{F,H\}_{PB}} \tag{11}$$

We can use this result in a rather curious way to re-derive Hamilton's equations of motion. We first observe that we can write the field ϕ as an integral:

$$\phi(\mathbf{x},t)=\int d^3\mathbf{x}'\,\phi\left(\mathbf{x}',t\right)\delta^3\left(\mathbf{x}-\mathbf{x}'\right) \tag{12}$$

This effectively defines an ordinary function ϕ as a functional depending on itself. In this case, both $\mathbf{x}$ and t are parameters that are present on both sides of the equation; it is the dummy variable $\mathbf{x}'$ that is the variable of integration.

Taking the variation on both sides, we get

$$\delta\phi\left(\mathbf{x},t\right)=\int d^{3}\mathbf{x}'\;\delta\phi\left(\mathbf{x}',t\right)\delta^{3}\left(\mathbf{x}-\mathbf{x}'\right) \tag{13}$$

[Be careful not to get the δs confused here: $\delta\phi$ is a variation of the function ϕ while δ^3 is the 3-d delta function.] Comparing this to the definition of the functional derivative

$$\delta F\left[\phi\right]\equiv\int d^{3}\mathbf{x}\frac{\delta F\left[\phi\right]}{\delta\phi\left(x\right)}\delta\phi\left(x\right) \tag{14}$$

we see that we have the functional derivative of ϕ with respect to itself:

$$\frac{\delta\phi\left(\mathbf{x},t\right)}{\delta\phi\left(\mathbf{x}',t\right)}=\delta^{3}\left(\mathbf{x}-\mathbf{x}'\right) \tag{15}$$

We could use the same argument on the conjugate momentum, so we also have

$$\frac{\delta\pi\left(\mathbf{x},t\right)}{\delta\pi\left(\mathbf{x}',t\right)}=\delta^{3}\left(\mathbf{x}-\mathbf{x}'\right) \tag{16}$$

Since ϕ and π are independent fields

$$\frac{\delta\pi\left(\mathbf{x},t\right)}{\delta\phi\left(\mathbf{x}',t\right)}=\frac{\delta\phi\left(\mathbf{x},t\right)}{\delta\pi\left(\mathbf{x}',t\right)}=0 \tag{17}$$

We can now use 11 to find the time derivatives of ϕ and π by treating them as functionals:

$$\begin{aligned}
\dot{\phi}\left(\mathbf{x},t\right) &= \left\{\phi\left(\mathbf{x},t\right),H\right\} && (18)\\
&= \int d^{3}\mathbf{x}'\left[\frac{\delta\phi\left(\mathbf{x},t\right)}{\delta\phi\left(\mathbf{x}',t\right)}\frac{\delta H\left(t\right)}{\delta\pi\left(\mathbf{x}',t\right)}-\frac{\delta\phi\left(\mathbf{x},t\right)}{\delta\pi\left(\mathbf{x}',t\right)}\frac{\delta H\left(t\right)}{\delta\phi\left(\mathbf{x}',t\right)}\right] && (19)\\
&= \int d^{3}\mathbf{x}'\left[\delta^{3}\left(\mathbf{x}-\mathbf{x}'\right)\frac{\delta H\left(t\right)}{\delta\pi\left(\mathbf{x}',t\right)}-0\right] && (20)\\
&= \frac{\delta H\left(t\right)}{\delta\pi\left(\mathbf{x},t\right)} && (21)
\end{aligned}$$

This gives the first Hamilton equation of motion 7. We can work out the second equation similarly:

$$\dot{\pi}(\mathbf{x},t) = \{\pi(\mathbf{x},t), H\} \tag{22}$$

$$= \int d^3\mathbf{x}' \left[\frac{\delta\pi(\mathbf{x},t)}{\delta\phi(\mathbf{x}',t)}\frac{\delta H(t)}{\delta\pi(\mathbf{x}',t)} - \frac{\delta\pi(\mathbf{x},t)}{\delta\pi(\mathbf{x}',t)}\frac{\delta H(t)}{\delta\phi(\mathbf{x}',t)}\right] \tag{23}$$

$$= \int d^3\mathbf{x}' \left[0 - \delta^3\left(\mathbf{x}-\mathbf{x}'\right)\frac{\delta H(t)}{\delta\phi(\mathbf{x}',t)}\right] \tag{24}$$

$$= -\frac{\delta H(t)}{\delta\phi(\mathbf{x},t)} \tag{25}$$

Finally, we can work out the Poisson brackets of the fields with each other, using the definition 10 and the results above.

$$\left\{\phi(\mathbf{x},t), \pi\left(\mathbf{x}',t\right)\right\}_{PB} \equiv \int d^3\mathbf{x}'' \left[\frac{\delta\phi(\mathbf{x},t)}{\delta\phi(\mathbf{x}'',t)}\frac{\delta\pi(\mathbf{x}',t)}{\delta\pi(\mathbf{x}'',t)} - \frac{\delta\phi(\mathbf{x},t)}{\delta\pi(\mathbf{x}'',t)}\frac{\delta\pi(\mathbf{x}',t)}{\delta\phi(\mathbf{x}'',t)}\right] \tag{26}$$

$$= \int d^3\mathbf{x}'' \left[\delta^3\left(\mathbf{x}-\mathbf{x}''\right)\delta^3\left(\mathbf{x}'-\mathbf{x}''\right) - 0\right] \tag{27}$$

$$= \delta^3\left(\mathbf{x}-\mathbf{x}'\right) \tag{28}$$

The other two Poisson brackets are zero because of 17:

$$\left\{\phi(\mathbf{x},t), \phi\left(\mathbf{x}',t\right)\right\}_{PB} \equiv \int d^3\mathbf{x}'' \left[\frac{\delta\phi(\mathbf{x},t)}{\delta\phi(\mathbf{x}'',t)}\frac{\delta\phi(\mathbf{x}',t)}{\delta\pi(\mathbf{x}'',t)} - \frac{\delta\phi(\mathbf{x},t)}{\delta\pi(\mathbf{x}'',t)}\frac{\delta\phi(\mathbf{x}',t)}{\delta\phi(\mathbf{x}'',t)}\right] \tag{29}$$

$$= \int d^3\mathbf{x}'' [0-0] \tag{30}$$

$$= 0 \tag{31}$$

$$\left\{\pi(\mathbf{x},t), \pi\left(\mathbf{x}',t\right)\right\}_{PB} \equiv \int d^3\mathbf{x}'' \left[\frac{\delta\pi(\mathbf{x},t)}{\delta\phi(\mathbf{x}'',t)}\frac{\delta\pi(\mathbf{x}',t)}{\delta\pi(\mathbf{x}'',t)} - \frac{\delta\pi(\mathbf{x},t)}{\delta\pi(\mathbf{x}'',t)}\frac{\delta\pi(\mathbf{x}',t)}{\delta\phi(\mathbf{x}'',t)}\right] \tag{32}$$

$$= \int d^3\mathbf{x}'' [0-0] \tag{33}$$

$$= 0 \tag{34}$$

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