

POISSON BRACKETS AND HAMILTON'S EQUATIONS OF MOTION

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Reference: W. Greiner & J. Reinhardt, *Field Quantization*, Springer-Verlag (1996), Chapter 2, Section 2.2.

Although I've looked at Poisson brackets before, it's worth going through G & R's treatment as it is a fair bit simpler and gives clearer results.

First, we need the time derivative of a functional. In the simplest case, a functional $F[\phi]$ depends on a function ϕ which in turn depends on an independent variable x . The functional itself does not depend on x , however, usually because F is defined as the integral of $\phi(x)$ over some range of x values, so the dependence on x disappears in the integration.

We can generalize things a bit by taking ϕ as a function of two variables, say x and t . If F is defined in the same way (say, as an integral of ϕ over x), then the variable t also appears in the functional, so we can write this as $F(t)$, which is

$$F(t) = \int dx g(\phi(x,t)) \quad (1)$$

where $g(\phi)$ is some function of ϕ . Since F now depends on t , we can take the derivative dF/dt which comes out to

$$\dot{F} \equiv \frac{dF}{dt} = \int dx \frac{dg}{d\phi} \frac{\partial \phi}{\partial t} = \int dx \frac{dg}{d\phi} \dot{\phi}(x,t) \quad (2)$$

As we've seen before, the functional derivative of F in this case is

$$\frac{\delta F(t)}{\delta \phi(y,t)} = \frac{dg(\phi(y,t))}{d\phi} \quad (3)$$

where the notation means that we evaluate the derivative on the RHS at the point (y,t) . Using this result, we can therefore write \dot{F} as

$$\dot{F}(t) = \int dx \frac{\delta F(t)}{\delta \phi(x,t)} \dot{\phi}(x,t) \quad (4)$$

We can generalize this to 4-d space time, so that x now indicates the four-vector $x = (\mathbf{x}, t)$, and the integral is over 3-d space:

$$\dot{F}(t) = \int d^3\mathbf{x} \frac{\delta F(t)}{\delta \phi(x)} \dot{\phi}(x) \quad (5)$$

Generalizing even further, we can make F a functional of two fields, ϕ and π , so we get

$$\dot{F}(t) = \int d^3\mathbf{x} \left[\frac{\delta F(t)}{\delta \phi(x)} \dot{\phi}(x) + \frac{\delta F(t)}{\delta \pi(x)} \dot{\pi}(x) \right] \quad (6)$$

Interpreting ϕ as the field and π as its conjugate momentum, we can now use Hamilton's equations of motion

$$\dot{\phi} = \frac{\delta H}{\delta \pi} \quad (7)$$

$$\dot{\pi} = -\frac{\delta H}{\delta \phi} \quad (8)$$

We get

$$\dot{F}(t) = \int d^3\mathbf{x} \left[\frac{\delta F(t)}{\delta \phi(x)} \frac{\delta H}{\delta \pi} - \frac{\delta F(t)}{\delta \pi(x)} \frac{\delta H}{\delta \phi} \right] \quad (9)$$

The quantity on the RHS is defined to be the Poisson bracket:

$$\{F, H\}_{PB} \equiv \int d^3\mathbf{x} \left[\frac{\delta F(t)}{\delta \phi(x)} \frac{\delta H}{\delta \pi} - \frac{\delta F(t)}{\delta \pi(x)} \frac{\delta H}{\delta \phi} \right] \quad (10)$$

We thus have the general result that the time derivative of a functional is equal to its Poisson bracket with the Hamiltonian:

$$\boxed{\dot{F}(t) = \{F, H\}_{PB}} \quad (11)$$

We can use this result in a rather curious way to re-derive Hamilton's equations of motion. We first observe that we can write the field ϕ as an integral:

$$\phi(\mathbf{x}, t) = \int d^3\mathbf{x}' \phi(\mathbf{x}', t) \delta^3(\mathbf{x} - \mathbf{x}') \quad (12)$$

This effectively defines an ordinary function ϕ as a functional depending on itself. In this case, both \mathbf{x} and t are parameters that are present on both sides of the equation; it is the dummy variable \mathbf{x}' that is the variable of integration.

Taking the variation on both sides, we get

$$\delta\phi(\mathbf{x}, t) = \int d^3\mathbf{x}' \delta\phi(\mathbf{x}', t) \delta^3(\mathbf{x} - \mathbf{x}') \quad (13)$$

[Be careful not to get the δ s confused here: $\delta\phi$ is a variation of the function ϕ while δ^3 is the 3-d delta function.] Comparing this to the definition of the functional derivative

$$\delta F[\phi] \equiv \int d^3\mathbf{x} \frac{\delta F[\phi]}{\delta\phi(\mathbf{x})} \delta\phi(\mathbf{x}) \quad (14)$$

we see that we have the functional derivative of ϕ with respect to itself:

$$\frac{\delta\phi(\mathbf{x},t)}{\delta\phi(\mathbf{x}',t)} = \delta^3(\mathbf{x} - \mathbf{x}') \quad (15)$$

We could use the same argument on the conjugate momentum, so we also have

$$\frac{\delta\pi(\mathbf{x},t)}{\delta\pi(\mathbf{x}',t)} = \delta^3(\mathbf{x} - \mathbf{x}') \quad (16)$$

Since ϕ and π are independent fields

$$\frac{\delta\pi(\mathbf{x},t)}{\delta\phi(\mathbf{x}',t)} = \frac{\delta\phi(\mathbf{x},t)}{\delta\pi(\mathbf{x}',t)} = 0 \quad (17)$$

We can now use 11 to find the time derivatives of ϕ and π by treating them as functionals:

$$\dot{\phi}(\mathbf{x},t) = \{\phi(\mathbf{x},t), H\} \quad (18)$$

$$= \int d^3\mathbf{x}' \left[\frac{\delta\phi(\mathbf{x},t)}{\delta\phi(\mathbf{x}',t)} \frac{\delta H(t)}{\delta\pi(\mathbf{x}',t)} - \frac{\delta\phi(\mathbf{x},t)}{\delta\pi(\mathbf{x}',t)} \frac{\delta H(t)}{\delta\phi(\mathbf{x}',t)} \right] \quad (19)$$

$$= \int d^3\mathbf{x}' \left[\delta^3(\mathbf{x} - \mathbf{x}') \frac{\delta H(t)}{\delta\pi(\mathbf{x}',t)} - 0 \right] \quad (20)$$

$$= \frac{\delta H(t)}{\delta\pi(\mathbf{x},t)} \quad (21)$$

This gives the first Hamilton equation of motion 7. We can work out the second equation similarly:

$$\dot{\pi}(\mathbf{x}, t) = \{\pi(\mathbf{x}, t), H\} \quad (22)$$

$$= \int d^3\mathbf{x}' \left[\frac{\delta\pi(\mathbf{x}, t)}{\delta\phi(\mathbf{x}', t)} \frac{\delta H(t)}{\delta\pi(\mathbf{x}', t)} - \frac{\delta\pi(\mathbf{x}, t)}{\delta\pi(\mathbf{x}', t)} \frac{\delta H(t)}{\delta\phi(\mathbf{x}', t)} \right] \quad (23)$$

$$= \int d^3\mathbf{x}' \left[0 - \delta^3(\mathbf{x} - \mathbf{x}') \frac{\delta H(t)}{\delta\phi(\mathbf{x}', t)} \right] \quad (24)$$

$$= - \frac{\delta H(t)}{\delta\phi(\mathbf{x}, t)} \quad (25)$$

Finally, we can work out the Poisson brackets of the fields with each other, using the definition 10 and the results above.

$$\{\phi(\mathbf{x}, t), \pi(\mathbf{x}', t)\}_{PB} \equiv \int d^3\mathbf{x}'' \left[\frac{\delta\phi(\mathbf{x}, t)}{\delta\phi(\mathbf{x}'', t)} \frac{\delta\pi(\mathbf{x}', t)}{\delta\pi(\mathbf{x}'', t)} - \frac{\delta\phi(\mathbf{x}, t)}{\delta\pi(\mathbf{x}'', t)} \frac{\delta\pi(\mathbf{x}', t)}{\delta\phi(\mathbf{x}'', t)} \right] \quad (26)$$

$$= \int d^3\mathbf{x}'' [\delta^3(\mathbf{x} - \mathbf{x}'') \delta^3(\mathbf{x}' - \mathbf{x}'') - 0] \quad (27)$$

$$= \delta^3(\mathbf{x} - \mathbf{x}') \quad (28)$$

The other two Poisson brackets are zero because of 17:

$$\{\phi(\mathbf{x}, t), \phi(\mathbf{x}', t)\}_{PB} \equiv \int d^3\mathbf{x}'' \left[\frac{\delta\phi(\mathbf{x}, t)}{\delta\phi(\mathbf{x}'', t)} \frac{\delta\phi(\mathbf{x}', t)}{\delta\pi(\mathbf{x}'', t)} - \frac{\delta\phi(\mathbf{x}, t)}{\delta\pi(\mathbf{x}'', t)} \frac{\delta\phi(\mathbf{x}', t)}{\delta\phi(\mathbf{x}'', t)} \right] \quad (29)$$

$$= \int d^3\mathbf{x}'' [0 - 0] \quad (30)$$

$$= 0 \quad (31)$$

$$\{\pi(\mathbf{x}, t), \pi(\mathbf{x}', t)\}_{PB} \equiv \int d^3\mathbf{x}'' \left[\frac{\delta\pi(\mathbf{x}, t)}{\delta\phi(\mathbf{x}'', t)} \frac{\delta\pi(\mathbf{x}', t)}{\delta\pi(\mathbf{x}'', t)} - \frac{\delta\pi(\mathbf{x}, t)}{\delta\pi(\mathbf{x}'', t)} \frac{\delta\pi(\mathbf{x}', t)}{\delta\phi(\mathbf{x}'', t)} \right] \quad (32)$$

$$= \int d^3\mathbf{x}'' [0 - 0] \quad (33)$$

$$= 0 \quad (34)$$

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