COORDINATE TRANSFORMATIONS IN CLASSICAL FIELD THEORY

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Reference: W. Greiner & J. Reinhardt, *Field Quantization*, Springer-Verlag (1996), Chapter 2, Section 2.4.

The various conservation laws of physics (energy, linear and angular momentum) can be derived from the invariance of a system under coordinate transformations. To prepare for *Noether's theorem*, which is a general theorem allowing us to derive these conservation laws, we need to consider how the fields themselves transform under coordinate transformations.

In what follows, we'll consider only infinitesimal transformations, and we define a general transformation as

$$x'_\mu = x_\mu + \delta x_\mu \tag{1}$$

Note that x_μ and x'_μ both refer to the same physical point in space; they simply represent two different coordinate systems referring to this same point.

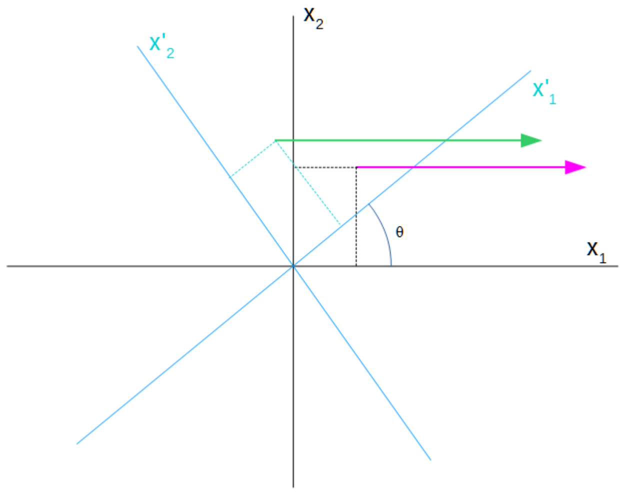
Under this transformation, the mathematical function describing the field will change as well, so we can write

$$\phi'_r\left(x'\right) = \phi_r\left(x\right) + \delta\phi_r\left(x\right) \tag{2}$$

where the subscript r labels which field we're talking about.

Again, $\phi'_r(x')$ and $\phi_r(x)$ both represent the same field at the same point in space-time; they are just expressed in different coordinate systems.

At this point, it's useful to have a look at a specific example. Suppose the field ϕ is a vector field in two dimensions (we'll drop the r subscript, as we're dealing with only one field). We'll see what happens if we rotate the coordinate system through an angle θ, as in the diagram, where the unprimed system is drawn in black and the primed system in blueo.



In the unprimed system, ϕ consists of horizontal vectors with a magnitude equal to their x_2 coordinate.

$$\phi(x) = \begin{bmatrix} x_2 \\ 0 \end{bmatrix} \tag{3}$$

Under a rotation, the coordinates transform according to

$$x' = \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1\cos\theta + x_2\sin\theta \\ -x_1\sin\theta + x_2\cos\theta \end{bmatrix} \tag{4}$$

Inverting the rotation gives

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = \begin{bmatrix} x'_1\cos\theta - x'_2\sin\theta \\ x'_1\sin\theta + x'_2\cos\theta \end{bmatrix} \tag{5}$$

For our example vector field 3, we have, by inserting 3 into 4 (that is, in 4 we set $x_1 = x_2$ and $x_2 = 0$):

$$\phi'(x') = \begin{bmatrix} x_2\cos\theta \\ -x_2\sin\theta \end{bmatrix} = \begin{bmatrix} x'_1\sin\theta\cos\theta + x'_2\cos^2\theta \\ -x'_1\sin^2\theta - x'_2\sin\theta\cos\theta \end{bmatrix} \tag{6}$$

As we can see from the diagram by looking at the magenta vector, the vector in the unprimed system is parallel to the x_1 axis, with length x_2 as given by 3. If we rotate the coordinate axes by the angle θ we get the primed system shown as the blue axes, and we can see that in that system, the magenta vector has a positive component in the x'_1 direction and a negative component in the x'_2 direction. However, the length of the vector remains the same in both systems, since the vector itself doesn't change when we simply rotate the coordinates. (We'll explain the green vector later on in this post.)

Since we'll deal primarily with infinitesimal transformations from now on, we'll do the rest of the analysis using that approximation. For the rotation example above, if θ is now an infinitesimal angle (I suppose I should write it as $\delta\theta$ but this just clutters up the notation, so just remember that θ is infinitesimal and all will be well.), then we have, to first order in θ, $\cos\theta = 1$ and $\sin\theta = \theta$, so for a general rotation

$$x' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} x_1 + x_2\theta \\ -x_1\theta + x_2 \end{bmatrix} \tag{7}$$

$$\delta x = x' - x = \begin{bmatrix} x_2\theta \\ -x_1\theta \end{bmatrix} \tag{8}$$

For the specific example above, to first order in θ

$$\phi'\left(x'\right) = \begin{bmatrix} x_2 \\ -x_2\theta \end{bmatrix} = \begin{bmatrix} x_1'\theta + x_2' \\ -x_2'\theta \end{bmatrix} \tag{9}$$

Plugging 3 and 9 into 2, we get

$$\delta\phi(x) = \phi'\left(x'\right) - \phi(x) = \begin{bmatrix} 0 \\ -x_2\theta \end{bmatrix} \tag{10}$$

Up to now, we've considered what happens at one specific point when the coordinate system is varied. The variation $\delta\phi(x)$ is the result of varying both the coordinate system and the effect this variation has on the form of the field expression. In practice, another kind of variation, called the *modified* or *total* variation is defined by

$$\tilde{\delta}\phi_r(x) \equiv \phi_r'(x) - \phi_r(x) \tag{11}$$

Note that the difference between $\tilde{\delta}\phi_r(x)$ and $\delta\phi_r(x)$ is that the ϕ_r' term is evaluated at x in the former and at x' in the latter. This notation is somewhat confusing, since in 2, both x' and x refer to the *same* point in the plane, while in the latter, the x in $\phi_r'(x)$ is a *different* point from the x in $\phi_r(x)$. We can illustrate this by looking again at the above diagram. The point x in the unprimed system is at around $(x_1, x_2) = (1, 2)$ (it's the location of the tail of the magenta vector, identified by the dotted black lines). The notation $\phi_r'(x)$ means that we insert the same numerical values for (x_1, x_2) into the function ϕ_r', that is, we set $(x_1', x_2') = (1, 2)$. This gives the location indicated by the tail of the green vector, as identified by the dotted blue lines. Since this location is higher up the x_2 axis than the magenta vector, the green vector is longer than the magenta vector, so that $\phi_r'(x)$ and $\phi_r(x)$ now refer to two *different* vectors. The quantity $\tilde{\delta}\phi_r(x)$ therefore measures the change in the field due solely to the transformation of the coordinates.

We can, nevertheless, derive a relation between $\tilde{\delta}\phi_r(x)$ and $\delta\phi_r(x)$. Starting from 11, we have

$$\begin{aligned}
\tilde{\delta}\phi_r(x) &= \phi_r'(x) - \phi_r(x) && (12)\\
&= \phi_r'(x) - \phi_r'(x') + \phi_r'(x') - \phi_r(x) && (13)\\
&= -\left(\phi_r'(x') - \phi_r'(x)\right) + \delta\phi_r(x) && (14)\\
&= \delta\phi_r(x) - \frac{\partial\phi_r'(x)}{\partial x_\mu}\delta x_\mu && (15)\\
&= \delta\phi_r(x) - \frac{\partial\phi_r(x)}{\partial x_\mu}\delta x_\mu && (16)
\end{aligned}$$

In the penultimate line, we replaced $\phi_r'(x') - \phi_r'(x)$ by its first order term in the Taylor expansion, and in the last line, we approximated $\phi_r'(x)$ by $\phi_r(x)$, again valid to first order.

As an example, we can apply this formula to the above vector field. Starting with 11, we have, using 9 and 3

$$\begin{aligned}
\tilde{\delta}\phi(x) &= \phi'(x) - \phi(x) && (17)\\
&= \begin{bmatrix} x_1\theta + x_2 \\ -x_2\theta \end{bmatrix} - \begin{bmatrix} x_2 \\ 0 \end{bmatrix} && (18)\\
&= \begin{bmatrix} x_1\theta \\ -x_2\theta \end{bmatrix} && (19)
\end{aligned}$$

Now we can check 16. From 3 we have

$$\begin{aligned}
\frac{\partial\phi(x)}{\partial x_1} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} && (20)\\
\frac{\partial\phi(x)}{\partial x_2} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} && (21)
\end{aligned}$$

From 8, we have

$$\frac{\partial\phi_r(x)}{\partial x_\mu}\delta x_\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\delta x_1 + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\delta x_2 \tag{22}$$
$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -x_1\theta \\ 0 \end{bmatrix} \tag{23}$$
$$= -\begin{bmatrix} x_1\theta \\ 0 \end{bmatrix} \tag{24}$$

Combining this with 10 we get

$$\tilde{\delta}\phi_r(x) = \delta\phi_r(x) - \frac{\partial\phi_r(x)}{\partial x_\mu}\delta x_\mu \tag{25}$$
$$= \begin{bmatrix} 0 \\ -x_2\theta \end{bmatrix} + \begin{bmatrix} x_1\theta \\ 0 \end{bmatrix} \tag{26}$$
$$= \begin{bmatrix} x_1\theta \\ -x_2\theta \end{bmatrix} \tag{27}$$

which agrees with 19.

Finally, we can note a couple of formulas concerning the derivative of the two variations $\tilde{\delta}\phi_r(x)$ and $\delta\phi_r(x)$. Since $\tilde{\delta}\phi_r(x)$ depends only on x (and not on x'), the derivative commutes with the variation:

$$\frac{\partial}{\partial x_\mu}\tilde{\delta}\phi_r(x) = \tilde{\delta}\left(\frac{\partial\phi_r(x)}{\partial x_\mu}\right) \tag{28}$$

The other variation $\delta\phi_r(x)$ is a bit trickier, since it involves x' as well as x. However, using the chain rule, we can find its derivative. I'll use the shorthand $\partial_\mu \equiv \partial/\partial x_\mu$ and $\partial'_\mu \equiv \partial/\partial x'_\mu$.

$$\partial_\mu(\delta\phi_r(x)) = \partial_\mu\phi'_r(x') - \partial_\mu\phi_r(x) \tag{29}$$
$$= \left[\partial'_\mu\phi'_r(x') - \partial_\mu\phi_r(x)\right] + \partial_\mu\phi'_r(x') - \partial'_\mu\phi'_r(x') \tag{30}$$
$$= \delta\left(\partial_\mu\phi_r(x)\right) + \left(\partial'_\nu\phi'_r(x')\right)\left(\partial_\mu x'^\nu\right) - \partial'_\mu\phi'_r(x') \tag{31}$$

We can now use 1 on the middle term:

$$\partial_\mu x'^\nu = \partial_\mu(x^\nu + \delta x^\nu) \tag{32}$$
$$= \delta^{\mu\nu} + \partial_\mu\delta x^\nu \tag{33}$$

Combining the last two terms, we get

$$\left(\partial_\nu^\prime\phi_r^\prime\left(x^\prime\right)\right)\left(\delta^{\mu\nu}+\partial_\mu\delta x^\nu\right)-\partial_\mu^\prime\phi_r^\prime\left(x^\prime\right)=\left(\partial_\nu^\prime\phi_r^\prime\left(x^\prime\right)\right)\partial_\mu\delta x^\nu \tag{34}$$

$$=\left(\partial_\nu\phi_r\left(x\right)\right)\partial_\mu\delta x^\nu \tag{35}$$

Again, the last step is valid to first order in the variations. Thus we have

$$\partial_\mu\left(\delta\phi_r\left(x\right)\right)=\delta\left(\partial_\mu\phi_r\left(x\right)\right)+\left(\partial_\nu\phi_r\left(x\right)\right)\partial_\mu\delta x^\nu \tag{36}$$

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