NOETHER'S THEOREM AND CONSERVATION LAWS

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Reference: W. Greiner & J. Reinhardt, *Field Quantization*, Springer-Verlag (1996), Chapter 2, Section 2.4.

We can now apply the formulas resulting from coordinate transformations to derive Noether's theorem, which states that for each coordinate transformation that leaves the physics of a system unchanged, there is a corresponding conserved quantity. More precisely, if we transform the coordinates according to

$$x'_\mu = x_\mu + \delta x_\mu \tag{1}$$

this results in a variation of the fields

$$\phi'_r\left(x'\right) = \phi_r(x) + \delta\phi_r(x) \tag{2}$$

We can insert these varied quantities into the Lagrangian density to get its variation:

$$\mathcal{L}'\left(x'\right) = \mathcal{L}(x) + \delta\mathcal{L}(x) \tag{3}$$

[Note that the Lagrangian density actually depends on the fields ϕ_r and their derivatives $\partial_\mu\phi_r$, both of which depend, in turn, on the coordinates x_μ, but having to write $\mathcal{L}(\phi_r(x), \partial_\mu\phi_r(x))$ everywhere would get very tedious, so we'll use $\mathcal{L}(x) = \mathcal{L}(\phi_r(x), \partial_\mu\phi_r(x))$ as a shorthand.] The function $\mathcal{L}'(x')$ is just $\mathcal{L}(x)$ with x replaced by x' and $\phi_r(x)$ replaced by $\phi'_r(x')$.

The mathematical interpretation of the phrase "the physics doesn't change" is expressed by requiring the action of the system to remain the same when $\mathcal{L}$ is varied. Using G & R's symbol W (rather than the more usual S) for the action, this requirement is

$$\delta W \equiv \int_{\Omega'} d^4x' \mathcal{L}'\left(x'\right) - \int_{\Omega} d^4x\, \mathcal{L}(x) = 0 \tag{4}$$

As explained earlier, both x' and x refer to the same point in spacetime, written in two different coordinate systems. The volume Ω' is the same volume as Ω, but written in the x' coordinate system.

From here, we can follow G & R's derivation of Noether's theorem, which I find somewhat easier to follow than the one in Peskin & Schroeder, which I looked at earlier. In order to understand what 4 is saying, we first need to express everything in terms of one coordinate system, which we'll take to be x. First, we look at the volume element d^4x' in the first integral. We can express this in terms of the volume element d^4x by using the Jacobian determinant, using 1 to calculate the derivatives.

$$d^4x' = \left|\frac{\partial\left(x'_\mu\right)}{\partial\left(x_\nu\right)}\right| \tag{5}$$

$$= \left|\begin{array}{cccc} 1+\frac{\partial\delta x_0}{\partial x_0} & \frac{\partial\delta x_0}{\partial x_1} & \frac{\partial\delta x_0}{\partial x_2} & \frac{\partial\delta x_0}{\partial x_3} \\ \frac{\partial\delta x_1}{\partial x_0} & 1+\frac{\partial\delta x_1}{\partial x_1} & \frac{\partial\delta x_1}{\partial x_2} & \frac{\partial\delta x_1}{\partial x_3} \\ \frac{\partial\delta x_2}{\partial x_0} & \frac{\partial\delta x_2}{\partial x_1} & 1+\frac{\partial\delta x_2}{\partial x_2} & \frac{\partial\delta x_2}{\partial x_3} \\ \frac{\partial\delta x_3}{\partial x_0} & \frac{\partial\delta x_3}{\partial x_1} & \frac{\partial\delta x_3}{\partial x_2} & 1+\frac{\partial\delta x_3}{\partial x_3} \end{array}\right| \tag{6}$$

Since we're considering only infinitesimal variations, we need to keep only up to first order terms in this determinant. If we expand the determinant about the first row, the first term is

$$\left(1+\frac{\partial\delta x_0}{\partial x_0}\right)\left|\begin{array}{ccc} 1+\frac{\partial\delta x_1}{\partial x_1} & \frac{\partial\delta x_1}{\partial x_2} & \frac{\partial\delta x_1}{\partial x_3} \\ \frac{\partial\delta x_2}{\partial x_1} & 1+\frac{\partial\delta x_2}{\partial x_2} & \frac{\partial\delta x_2}{\partial x_3} \\ \frac{\partial\delta x_3}{\partial x_1} & \frac{\partial\delta x_3}{\partial x_2} & 1+\frac{\partial\delta x_3}{\partial x_3} \end{array}\right| = \tag{7}$$

$$\left(1+\frac{\partial\delta x_0}{\partial x_0}\right)\left[\left(1+\frac{\partial\delta x_1}{\partial x_1}\right)\left(\left(1+\frac{\partial\delta x_2}{\partial x_2}\right)\left(1+\frac{\partial\delta x_3}{\partial x_3}\right)-\frac{\partial\delta x_2}{\partial x_3}\frac{\partial\delta x_3}{\partial x_2}\right)+\ldots\right] = \tag{8}$$

$$1+\frac{\partial\delta x_\mu}{\partial x_\mu}+\ldots \tag{9}$$

In the second line, all the terms represented by the ... are of second or higher order in δx_μ so can be omitted from the final result. Note that we're summing over μ in the last line. All terms arising from the remaining 3 terms in the expansion about the first row of 6 are also of second or higher order, so the final result, valid to first order, is

$$d^4x' = \left(1+\frac{\partial\delta x_\mu}{\partial x_\mu}\right)d^4x \tag{10}$$

So much for the volume element. The only remaining task is to express $\mathcal{L}'\left(x'\right)$ in the x coordinate system. To do this, we can use 3

$$\delta W = \int_{\Omega'} d^4x' \left[\delta\mathcal{L}(x) + \mathcal{L}(x)\right] - \int_{\Omega} d^4x\, \mathcal{L}(x) \tag{11}$$
$$= \int_{\Omega} d^4x \left[\left(1 + \frac{\partial \delta x_\mu}{\partial x_\mu}\right)(\delta\mathcal{L}(x) + \mathcal{L}(x)) - \mathcal{L}(x)\right] \tag{12}$$
$$= \int_{\Omega} d^4x \left[\left(1 + \frac{\partial \delta x_\mu}{\partial x_\mu}\right)\delta\mathcal{L}(x) + \frac{\partial \delta x_\mu}{\partial x_\mu}\mathcal{L}(x)\right] \tag{13}$$
$$= \int_{\Omega} d^4x\, \delta\mathcal{L}(x) + \int_{\Omega} d^4x\, \frac{\partial \delta x_\mu}{\partial x_\mu}\mathcal{L}(x) \tag{14}$$

where in the last line, we saved terms up to first order only. In the second line, we can replace the volume of integration Ω' by Ω in all integrals, since we've changed the integration variable from x' to x, and Ω and Ω' both represent the same volume, as mentioned above.

Now we can use the total variation, which is

$$\tilde{\delta}\mathcal{L}(x) = \delta\mathcal{L}(x) - \frac{\partial\mathcal{L}(x)}{\partial x_\mu}\delta x_\mu \tag{15}$$

We get

$$\delta W = \int_{\Omega} d^4x\, \left[\tilde{\delta}\mathcal{L}(x) + \frac{\partial\mathcal{L}(x)}{\partial x_\mu}\delta x_\mu\right] + \int_{\Omega} d^4x\, \frac{\partial \delta x_\mu}{\partial x_\mu}\mathcal{L}(x) \tag{16}$$
$$= \int_{\Omega} d^4x\, \left[\tilde{\delta}\mathcal{L}(x) + \partial^\mu\left(\mathcal{L}(x)\,\delta x_\mu\right)\right] \tag{17}$$

using the product rule (backwards) in the last line.

Now, remembering that $\mathcal{L}(x) = \mathcal{L}(\phi_r(x), \partial_\mu\phi_r(x))$, we can use the chain rule to expand the total variation of $\mathcal{L}$:

$$\tilde{\delta}\mathcal{L}(x) \quad = \quad \frac{\partial\mathcal{L}(x)}{\partial\phi_r}\tilde{\delta}\phi_r(x) + \frac{\partial\mathcal{L}(x)}{\partial(\partial^\mu\phi_r)}\tilde{\delta}\left(\partial^\mu\phi_r(x)\right) \tag{18}$$

We can now add and subtract the same term to the RHS (equivalent to adding zero) to get

$$\tilde{\delta}\mathcal{L}(x) = \left[\frac{\partial\mathcal{L}(x)}{\partial\phi_r}\tilde{\delta}\phi_r(x) - \partial^\mu\left(\frac{\partial\mathcal{L}(x)}{\partial(\partial^\mu\phi_r)}\right)\tilde{\delta}\phi_r(x)\right] \tag{19}$$
$$+\,\partial^\mu\left(\frac{\partial\mathcal{L}(x)}{\partial(\partial^\mu\phi_r)}\right)\tilde{\delta}\phi_r(x) + \frac{\partial\mathcal{L}(x)}{\partial(\partial^\mu\phi_r)}\tilde{\delta}\left(\partial^\mu\phi_r(x)\right) \tag{20}$$

As we saw earlier, the total variation operation $\tilde{\delta}$ commutes with differentiation with respect to x_μ so we can interchange the $\tilde{\delta}$ and ∂^μ in the last term to get

$$\tilde{\delta}\mathcal{L}(x) = \left[\frac{\partial\mathcal{L}(x)}{\partial\phi_r}\tilde{\delta}\phi_r(x) - \partial^\mu\left(\frac{\partial\mathcal{L}(x)}{\partial(\partial^\mu\phi_r)}\right)\tilde{\delta}\phi_r(x)\right] \tag{21}$$

$$+\partial^\mu\left(\frac{\partial\mathcal{L}(x)}{\partial(\partial^\mu\phi_r)}\right)\tilde{\delta}\phi_r(x) + \frac{\partial\mathcal{L}(x)}{\partial(\partial^\mu\phi_r)}\partial^\mu\left(\tilde{\delta}\phi_r(x)\right) \tag{22}$$

We can now use the reverse product rule on the last two terms to get

$$\tilde{\delta}\mathcal{L}(x) = \left[\frac{\partial\mathcal{L}(x)}{\partial\phi_r} - \partial^\mu\left(\frac{\partial\mathcal{L}(x)}{\partial(\partial^\mu\phi_r)}\right)\right]\tilde{\delta}\phi_r(x) \tag{23}$$

$$+\partial^\mu\left(\frac{\partial\mathcal{L}(x)}{\partial(\partial^\mu\phi_r)}\tilde{\delta}\phi_r(x)\right) \tag{24}$$

The term in square brackets is just the Euler-Lagrange equation and is zero if the fields ϕ_r satisfy the equations of motion:

$$\frac{\partial\mathcal{L}(x)}{\partial\phi_r} - \partial^\mu\left(\frac{\partial\mathcal{L}(x)}{\partial(\partial^\mu\phi_r)}\right) = 0 \tag{25}$$

We can therefore insert 24 back into 17 to get

$$\delta W = \int_\Omega d^4x\,\left[\partial^\mu\left(\frac{\partial\mathcal{L}(x)}{\partial(\partial^\mu\phi_r)}\tilde{\delta}\phi_r(x) + \mathcal{L}(x)\,\delta x_\mu\right)\right] \tag{26}$$

$$= \int_\Omega d^4x\,\left[\partial^\mu\left(\frac{\partial\mathcal{L}(x)}{\partial(\partial^\mu\phi_r)}\left(\delta\phi_r(x) - \partial^\nu\phi_r\delta x_\nu\right) + \mathcal{L}(x)\,\delta x_\mu\right)\right] \tag{27}$$

The requirement that $\delta W = 0$ must mean that the integrand is zero, since the volume Ω over which the integration is done is arbitrary. Thus we get

$$\partial^\mu\left(\frac{\partial\mathcal{L}(x)}{\partial(\partial^\mu\phi_r)}\left(\delta\phi_r(x) - \partial^\nu\phi_r\delta x_\nu\right) + \mathcal{L}(x)\,\delta x_\mu\right) = 0 \tag{28}$$

To see what this means, we can define the function f as

$$f_\mu(x) \equiv \frac{\partial\mathcal{L}(x)}{\partial(\partial^\mu\phi_r)}\left(\delta\phi_r(x) - \partial^\nu\phi_r\delta x_\nu\right) + \mathcal{L}(x)\,\delta x_\mu \tag{29}$$

so that

$$\partial^\mu f_\mu(x) = 0 \tag{30}$$

If we integrate this over 3-d space and use Gauss's theorem to convert the integral of a divergence to a surface integral, we have

$$\int_V d^3x\,\partial^\mu f_\mu(x) = \int_V d^3x\,\partial^0 f_0(x) + \int_V d^3x \nabla\cdot\mathbf{f}(x) \tag{31}$$

$$= \frac{d}{dx_0}\int_V d^3x\; f_0(x) + \int_S d\mathbf{a}\cdot\mathbf{f}(x) \tag{32}$$

$$= \frac{d}{dx_0}\int_V d^3x\; f_0(x) \tag{33}$$

where we make the usual assumption in the second line that $\mathbf{f}(x) \to 0$ fast enough at infinity that the surface integral is zero. However, the requirement 30 implies that the result of this volume integral must be zero as well, so that

$$\frac{d}{dx_0}\int_V d^3x\; f_0(x) = 0 \tag{34}$$

This implies that $f_0(x)$ is a conserved quantity, as its volume integral is constant over time. This is *Noether's theorem*, which we can state as:

A continuous symmetry transformation (given by 1 and 2) that leaves the physics unchanged (that is, there is no change in the action integral 4) leads to a conservation law, with the conserved quantity G given by

$$G \equiv \int_V d^3x\; f_0(x) \tag{35}$$

$$= \int_V d^3x\,\left(\frac{\partial\mathcal{L}(x)}{\partial(\partial^0\phi_r)}\left(\delta\phi_r(x) - \partial^\nu\phi_r\delta x_\nu\right) + \mathcal{L}(x)\,\delta x_0\right) \tag{36}$$

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