NOETHER'S THEOREM AND CONSERVATION OF ENERGY AND MOMENTUM

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Reference: W. Greiner & J. Reinhardt, *Field Quantization*, Springer-Verlag (1996), Chapter 2, Section 2.4.

An important example of Noether's theorem is the conservation of energy and momentum as consequences of the invariance of the action under coordinate translation in spacetime. Noether's theorem applies to the situation where we transform the coordinates according to

$$x'_{\mu}=x_{\mu}+\delta x_{\mu} \tag{1}$$

resulting in a variation of the fields

$$\phi'_{r}\left(x'\right)=\phi_{r}\left(x\right)+\delta\phi_{r}\left(x\right) \tag{2}$$

If this variation in coordinates and fields leaves the action integral unchanged, Noether's theorem says that the following condition must be satisfied:

$$\partial^{\mu}\left(\frac{\partial\mathcal{L}\left(x\right)}{\partial\left(\partial^{\mu}\phi_{r}\right)}\left(\delta\phi_{r}\left(x\right)-\partial_{\nu}\phi_{r}\delta x^{\nu}\right)+\mathcal{L}\left(x\right)\delta x_{\mu}\right)=0 \tag{3}$$

By integrating this over 3-d space and using Gauss's law, we find a conserved quantity G, given by

$$G\equiv\int_{V}d^{3}x\,\partial^{\mu}\left(\frac{\partial\mathcal{L}\left(x\right)}{\partial\left(\partial^{0}\phi_{r}\right)}\left(\delta\phi_{r}\left(x\right)-\partial_{\nu}\phi_{r}\delta x^{\nu}\right)+\mathcal{L}\left(x\right)\delta x_{0}\right) \tag{4}$$

Suppose we consider a translation in spacetime, so that the coordinates transform according to

$$x'_{\mu}=x_{\mu}+\epsilon_{\mu} \tag{5}$$

where the ϵ_{μ}s are infinitesimal (and independent) constants. That is, we're free to vary any (or all) of the coordinates by some infinitesimal amount. In particular, we can choose to make only one of the ϵ_{μ} variations non-zero. For example, we might choose ϵ_{0} to be non-zero with the remaining three $\epsilon_{i}=0$, which amounts to a translation in time but not in position.

Such a translation means that we perform the same experiment (the same 'physics') at a different time and/or at a different place, and we require that we get the same result under all such translations. Note that this does *not* mean that the behaviour of a system is independent of time or space. Rather, what it is saying is that if we imagine that the only thing that exists in the universe is the physical system we're studying, it shouldn't matter if we move the system to some other location, or start the experiment at an earlier or later time; in all cases we should observe the same behaviour. The system might evolve to different states as time passes, but the time-dependence of the system will be the same, as measured from the starting point we have chosen.

In terms of the fields, this amounts to saying that the fields will have exactly the same form when expressed in terms of the translation coordinates, that is

$$\phi_r'\left(x'\right)=\phi_r\left(x\right) \tag{6}$$

[Recall from our earlier discussion that x' and x both refer to the same point, but written in different coordinate systems. Under a translation, the value of a scalar field remains the same, as does a vector field, since all we've done is move the coordinate axes parallel to themselves. This is different from a rotation of the coordinates, under which a vector *does* changes its components in the new coordinate system (although its length remains unchanged).]

Thus we have

$$\delta\phi_r\left(x\right)=0 \tag{7}$$

and from 3

$$\partial^{\mu}\left(\frac{\partial\mathcal{L}\left(x\right)}{\partial\left(\partial^{\mu}\phi_r\right)}\partial_{\nu}\phi_r-g_{\mu\nu}\mathcal{L}\left(x\right)\right)\epsilon^{\nu}=0 \tag{8}$$

where we've used $\delta x_{\nu}=\epsilon_{\nu}$ and used the metric tensor $g_{\mu\nu}$ to lower the index: $\epsilon_{\mu}=g_{\mu\nu}\epsilon^{\nu}$. Since the ϵ_{ν} are arbitrary, we must have

$$\partial^{\mu}\left(\frac{\partial\mathcal{L}\left(x\right)}{\partial\left(\partial^{\mu}\phi_r\right)}\partial_{\nu}\phi_r-g_{\mu\nu}\mathcal{L}\left(x\right)\right)=0 \tag{9}$$

for each value of $\nu=0,1,2,3$ separately. Thus we get four conservation laws.

For $\nu=0$ we can apply the same procedure that was used to derive 4 from 3. That is, we integrate over 3-d space and use Gauss's law:

$$\int_V d^3x\,\partial^i\left(\frac{\partial\mathcal{L}(x)}{\partial(\partial^i\phi_r)}\partial_0\phi_r - g_{i0}\mathcal{L}(x)\right) = \partial^0\int_V d^3x\,\left(\frac{\partial\mathcal{L}(x)}{\partial(\partial^0\phi_r)}\partial_0\phi_r - g_{00}\mathcal{L}(x)\right) \tag{10}$$

On the LHS, the index i runs over the spatial indexes 1,2 and 3, and we've set $\nu = 0$ on both sides. The integral on the LHS is a divergence, so we use Gauss's law to convert this to a surface integral and extend the surface to infinity, requiring the integrand to go to zero fast enough that the integral is zero in the limit. We then get that

$$\partial^0\int_V d^3x\,\left(\frac{\partial\mathcal{L}(x)}{\partial(\partial^0\phi_r)}\partial_0\phi_r - g_{00}\mathcal{L}(x)\right) = 0 \tag{11}$$

so that the integral is a conserved quantity (it has zero time derivative).

Comparing this with Hamilton's equations of motion, we have the conjugate momentum density

$$\pi = \frac{\partial\mathcal{L}(x)}{\partial(\partial^0\phi_r)} \tag{12}$$

so the integrand of 11 becomes Hamilton's equation for the Hamiltonian density (using $g_{00} = 1$ in flat space):

$$\frac{\partial\mathcal{L}(x)}{\partial(\partial^0\phi_r)}\partial_0\phi_r - g_{00}\mathcal{L}(x) = \pi_r\dot{\phi}_r - \mathcal{L} = \mathcal{H} \tag{13}$$

Since $\mathcal{H}$ is the energy density, 11 says that the total energy of the system is constant in time, so energy is conserved.

We can repeat the procedure for the other three values of ν to get

$$\partial^0\int_V d^3x\,\left(\frac{\partial\mathcal{L}(x)}{\partial(\partial^0\phi_r)}\partial_i\phi_r - g_{0i}\mathcal{L}(x)\right) = 0 \tag{14}$$

where the index $i = 1,2,3$.

Since $g_{0i} = 0$ in flat space, the integrand reduces to

$$p_i = \frac{\partial\mathcal{L}(x)}{\partial(\partial^0\phi_r)}\partial_i\phi_r = \pi_r\frac{\partial\phi_r}{\partial x^i} \tag{15}$$

As we've seen earlier, we can interpret this quantity as the physical momentum density, so 14 says that each component of the total physical momentum is conserved. Thus requiring a physical system to be invariant under translation in spacetime results in the laws of conservation of energy and linear momentum.

Going back to 9, the general conservation law says that

$$\partial^\mu T_{\mu\nu} = 0 \tag{16}$$

where

$$T_{\mu\nu}\equiv\frac{\partial\mathcal{L}\left(x\right)}{\partial\left(\partial^{\mu}\phi_{r}\right)}\partial_{\nu}\phi_{r}-g_{\mu\nu}\mathcal{L}\left(x\right) \tag{17}$$

is the *energy-momentum tensor* and is defined for all values of μ and ν in the range 0,1,2,3. [G & R use the symbol $\Theta_{\mu\nu}$ for this tensor, but as it's the same as the stress-energy tensor, we'll try to keep the notation consistent at this point.]

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