LORENTZ TRANSFORMATIONS AS ROTATIONS

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References: W. Greiner & J. Reinhardt, *Field Quantization*, Springer-Verlag (1996), Chapter 2, Section 2.4.

Arthur Jaffe, *Lorentz transformations, rotations and boosts*, online notes available (at time of writing, Sep 2016) here.

Before we apply Noether's theorem to Lorentz transformations, we need to take a step back and look at a generalized version of the Lorentz transformation. Most introductory treatments of special relativity derive the Lorentz transformation as the transformation between two inertial frames that are moving at some constant velocity with respect to each other. This form of the transformations allows us to derive the usual consequences of special relativity such as length contraction and time dilation. However, it's useful to look at a Lorentz transformation is a more general way.

The idea is to define a Lorentz transformation as any transformation that leaves the magnitude of all four-vectors x unchanged, where this magnitude is defined using the usual flat space metric $g^{\mu\nu}$ so that

$$x^2 = x_\mu x^\mu = g^{\mu\nu} x_\mu x_\nu = x_0^2 - x_1^2 - x_2^2 - x_3^2 \tag{1}$$

The flat space (Minkowski) metric is

$$g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{2}$$

We know that the traditional Lorentz transformation between two inertial frames in relative motion satisfies this condition, but in fact a rotation of the coordinate system in 3-d space (leaving the time coordinate unchanged) also satisfies this condition, so a Lorentz transformation defined in this more general way includes more transformations than the traditional one.

We can define this general transformation in terms of a 4×4 matrix Λ, so that a four-vector x transforms to another vector x' according to

$$x' = \Lambda x \tag{3}$$

We can define the scalar product of two 4-vectors using the notation

$$\langle x,y\rangle \equiv \sum_{i=0}^{3} x_i y_i \tag{4}$$

The scalar product in flat space using the Minkowski metric g is therefore

$$\langle x,gy\rangle = g^{\mu\nu}x_\mu y_\nu = x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3 \tag{5}$$

In matrix notation, in which x and y are column vectors, this is

$$\langle x,gy\rangle = x^T g y \tag{6}$$

In this way, the condition that Λ leaves the magnitude unchanged is

$$\langle \Lambda x, g\Lambda x\rangle = \langle x, gx\rangle \tag{7}$$

for all x. In matrix notation, this is

$$(\Lambda x)^T g\Lambda x = x^T \Lambda^T g\Lambda x = x^T gx \tag{8}$$

from which we get one condition on Λ:

$$\Lambda^T g\Lambda = g \tag{9}$$

[Note that Jaffe uses a superscript tr to indicate a matrix transpose; I find this confusing as tr usually means the trace of a matrix, and a superscript T is more usual for the transpose.]

Because both sides of 9 refer to a symmetric matrix (on the LHS, $\left(\Lambda^T g\Lambda\right)^T = \Lambda^T g^T \left(\Lambda^T\right)^T = \Lambda^T g\Lambda$), this equation gives 10 independent equations for the elements of Λ, so the number of parameters that can be specified arbitrarily is $4\times 4 - 10 = 6$.

The set $\mathcal{L}$ of all Lorentz transformations forms a group under matrix multiplication, known as the *Lorentz group*. We can demonstrate this by showing that the four group properties are satisfied.

First, completeness. If we perform two transformations in succession on a 4-vector x then we get $x' = \Lambda_2\Lambda_1 x$. The compound transformation satisfies 9:

$$\begin{aligned}
(\Lambda_2\Lambda_1)^T g\Lambda_2\Lambda_1 &= \Lambda_1^T\Lambda_2^T g\Lambda_2\Lambda_1 && (10)\\
&= \Lambda_1^T g\Lambda_1 && (11)\\
&= g && (12)
\end{aligned}$$

Thus the group is closed under multiplication.

Second, associativity is automatically satisfied as matrix multiplication is associative.

An identity element exists in the form of the identity matrix I, which is itself a Lorentz transformation as it satisfies 9.

Finally, we need to show that every matrix Λ has an inverse that is also part of the set $\mathcal{L}$. Taking the determinant of 9 we have

$$\begin{aligned}
\det\left(\Lambda^T g\Lambda\right) &= \left(\det\Lambda^T\right)(\det g)(\det\Lambda) && (13)\\
&= (\det\Lambda)(\det g)(\det\Lambda) && (14)\\
&= -(\det\Lambda)^2 && (15)
\end{aligned}$$

since $\det g = -1$ from 2. From the RHS of 9, this must equal $\det g = -1$ so we have

$$\begin{aligned}
-(\det\Lambda)^2 &= -1 && (16)\\
\det\Lambda &= \pm 1 && (17)
\end{aligned}$$

From a basic theorem in matrix algebra, any matrix with a non-zero determinant has an inverse, so Λ^{-1} exists. To show that Λ^{-1} is a Lorentz transformation, we can take the inverse of 9 and use the fact that $g^{-1} = g$:

$$\begin{aligned}
\left(\Lambda^T g\Lambda\right)^{-1} &= g^{-1} = g && (18)\\
&= \Lambda^{-1} g\left(\Lambda^T\right)^{-1} && (19)\\
&= \Lambda^{-1} g\left(\Lambda^{-1}\right)^T && (20)
\end{aligned}$$

since the inverse and transpose operations commute (another basic theorem in matrix algebra). Therefore Λ^{-1} is also a valid Lorentz transformation.

We can also see that Λ^T is a valid transformation by left-multiplying by Λ and right-multiplying by Λ^T:

$$\begin{aligned}
g &= \Lambda^{-1} g\left(\Lambda^{-1}\right)^T && (21)\\
\Lambda g\Lambda^T &= \left(\Lambda\Lambda^{-1}\right) g\left(\Lambda^{-1}\right)^T \Lambda^T && (22)\\
&= g && (23)
\end{aligned}$$

We need one more property of Λ concerning the element Λ_{00}. Again starting from 9, the 00 component of the RHS is $g_{00} = 1$, and writing out the 00 component of the LHS explicitly we have

$$\left[\Lambda^T g\Lambda\right]_{00} = \Lambda_{00}^2 - \sum_{i=1}^{3}\Lambda_{i0}^2 = 1 \qquad (24)$$

This gives

$$\Lambda_{00} = \pm\sqrt{1 + \sum_{i=1}^{3} \Lambda_{i0}^2} \tag{25}$$

Thus either $\Lambda_{00} \geq 1$ or $\Lambda_{00} \leq -1$.

From the determinant and Λ_{00}, we can classify a particular transformation matrix Λ as being in one of four so-called *connected components*. Jaffe spells out in detail the proof that these four components are disjoint, that is, we can't define some parameter s that can be varied continuously to move a matrix Λ from one connected component to another connected component. The notation $\mathcal{L}_+^\uparrow$ indicates the set of matrices with $\det\Lambda = +1$ (indicated by the + subscript) and $\Lambda_{00} \geq 1$ (indicated by the $\uparrow$ superscript). The other three connected components are $\mathcal{L}_-^\uparrow$ ($\det\Lambda = -1$, $\Lambda_{00} \geq 1$); $\mathcal{L}_+^\downarrow$ ($\det\Lambda = +1$, $\Lambda_{00} \leq 1$); and $\mathcal{L}_-^\downarrow$ ($\det\Lambda = -1$, $\Lambda_{00} \leq 1$). Not all of these subsets of $\mathcal{L}$ form groups, as some of them are not closed under multiplication.

If $\det\Lambda = +1$, Λ is called *proper*, and if$\det\Lambda = -1$, Λ is called *improper.* If $\Lambda_{00} \geq +1$, Λ is *orthochronous*, and if$\Lambda_{00} \leq -1$, Λ is *non-orthochronous.* From here on, we'll consider only proper orthochronous transformations, that is, the connected component $\mathcal{L}_+^\uparrow$.

Members of $\mathcal{L}_+^\uparrow$ can be subdivided again into two types: *pure rotations* and *pure boosts.* A pure rotation is a rotation (about the origin) in 3-d space, leaving the time coordinate unchanged. That is, $\Lambda_{00} = +1$. Such a transformation can be written as

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & \mathcal{R} \end{bmatrix} \tag{26}$$

where $\mathcal{R}$ is a 3×3 matrix, and the 0s represent 3 zero components in the top row and first column. We know that the off-diagonal elements in the first column must be zero, since if $\Lambda_{00} = +1$, we have from 25 that

$$\sum_{i=1}^{3} \Lambda_{i0}^2 = 0 \tag{27}$$

Since Λ^T must also be a valid transformation, this gives the analogous equation

$$\sum_{i=1}^{3} \Lambda_{0i}^2 = 0 \tag{28}$$

Thus the off-diagonal elements of the top row of Λ are also zero.

Since $\det\Lambda = 1$, we must have $\det\mathcal{R} = 1$. From 9, $\mathcal{R}$ must also be an orthogonal matrix, that is, its rows must be mutually orthogonal (as must its columns). For example, if we pick the 2,3 element in the product 9, we have

$$\left[\Lambda^T g\Lambda\right]_{23} = g_{23} = 0 \tag{29}$$

$$= -\sum_{i=1}^{3}\Lambda_{i2}\Lambda_{i3} \tag{30}$$

Thus columns 2 and 3 must be orthogonal.

These matrices form a group known as $SO(3)$, the group of real, orthogonal, 3×3 matrices with $\det\mathcal{R} = +1$. A familiar example is a rotation by an angle θ about the z axis, for which

$$\mathcal{R} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{31}$$

giving the full transformation matrix as

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{32}$$

In general, a rotation can be about any axis through the origin, in which case $\mathcal{R}$ gets more complicated, but the idea is the same.

We've already seen that a pure boost, that is, a transformation into a second inertial frame moving at some constant velocity in a given direction relative to the first frame, can be written as a rotation, if we use hyperbolic functions instead of trig functions. In this case $\Lambda_{00} > +1$. The standard situation from introductory special relativity is that of from S' moving along the x_1 axis at some constant speed β. If we define

$$\cosh\chi \equiv \gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{33}$$

$$\sinh\chi \equiv \beta\gamma = \frac{\beta}{\sqrt{1-\beta^2}} \tag{34}$$

then the transformation is

$$\Lambda=\left[\begin{array}{cccc}\cosh\chi & \sinh\chi & 0 & 0\\ \sinh\chi & \cosh\chi & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right] \tag{35}$$

This has determinant +1 since $\cosh^2\chi-\sinh^2\chi=1$. We can verify by direct substitution that 9 is satisfied.

It turns out that all proper, orthochronous Lorentz transformations can be written as the product of a pure rotation and a pure boost, that is

$$\Lambda=BR \tag{36}$$

where the pure rotation R is applied *first*, followed by a pure boost B. (Jaffe doesn't prove this at this point; we'll return to this later.)

PINGBACKS

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Pingback: Lorentz transformations and the special linear group SL(2,C)

Pingback: Lorentz transformation as product of a pure boost and pure rotation

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