LORENTZ TRANSFORMATIONS AND THE SPECIAL LINEAR GROUP SL(2,C)

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References: W. Greiner & J. Reinhardt, *Field Quantization*, Springer-Verlag (1996), Chapter 2, Section 2.4.

Arthur Jaffe, *Lorentz transformations, rotations and boosts*, online notes available (at time of writing, Sep 2016) here.

Continuing our examination of general Lorentz transformations, we start off with the representation of a spacetime 4-vector as a 2×2 complex Hermitian matrix:

$$\widehat{x}\equiv\left[\begin{array}{cc} x_{0}+x_{3} & x_{1}-ix_{2}\\ x_{1}+ix_{2} & x_{0}-x_{3} \end{array}\right] \tag{1}$$

Our ultimate goal is to show that any Lorentz transformation can be represented as the product of a pure rotation R and a pure boost B: $\Lambda=RB$. The step shown in this post may look like little more than an exercise in matrix algebra, but be patient; it takes a while to get to our final goal.

We start by looking at the matrices belonging to the special linear group $SL(2,\mathbb{C})$, which consists of 2×2 matrices containing general complex numbers as elements, and with determinant 1. Each matrix $A\in SL(2,\mathbb{C})$ can be used to define a linear transformation of the Hermitian matrix 1:

$$\widehat{x}^{\prime}=A\widehat{x}A^{\dagger} \tag{2}$$

Because the determinant of a product is equal to the product of the determinants, and $\det A=\det A^{\dagger}=1$, $\det\widehat{x}^{\prime}=\det\widehat{x}=x_{\mu}x^{\mu}$. Thus such a transformation leaves the 4-vector length unchanged, so qualifies as a Lorentz transformation. Also, as a general complex 2×2 matrix contains 4 elements, each with a real and imaginary part, there are 8 parameters. The condition $\det A=1$ provides 2 constraints (one on the real part and one on the imaginary part), leaving 6 independent parameters, which is the same as the number of free parametersin a general Lorentz transformation.

We can give a more detailed proof that A provides a Lorentz transformation as follows. Suppose we start with two matrices $A,B\in SL(2,\mathbb{C})$ and define a transformation

$$\widehat{x}^{\prime}=A\widehat{x}B \tag{3}$$

[Remember that the hats on $\widehat{x}$ and $\widehat{x}'$ mean that we're considering the 2×2 matrix version 1 of the 4-vectors x and x'.] The transformed matrix $\widehat{x}'$ must be Hermitian for all $\widehat{x}$, so we must have

$$\begin{aligned}
\left(A\widehat{x}B\right)^{\dagger} &= A\widehat{x}B &(4)\\
&= B^{\dagger}\widehat{x}A^{\dagger} &(5)
\end{aligned}$$

We now left-multiply by $\left(B^{\dagger}\right)^{-1}$ and right-multiply by B^{-1} to get

$$\left(B^{\dagger}\right)^{-1}A\widehat{x}=\widehat{x}A^{\dagger}B^{-1} \tag{6}$$

But we also have

$$\left(B^{\dagger}\right)^{-1}A=\left(A^{\dagger}B^{-1}\right)^{\dagger} \tag{7}$$

so the matrix

$$T\equiv\left(B^{\dagger}\right)^{-1}A \tag{8}$$

is Hermitian. We can therefore write 6 as

$$T\widehat{x}=\widehat{x}T^{\dagger}=\widehat{x}T \tag{9}$$

so T commutes with $\widehat{x}$ for all $\widehat{x}$.

Now we can choose $x=\sigma_2$ and then $x=\sigma_3$, where the σ_is are two of the Pauli matrices which we showed (together with the identity matrix σ_0) form a basis for the space of 2×2 Hermitian matrices. Now we've seen thatσ_2 and σ_3 also form an irreducible set, and we saw that any matrix T that commutes with all the members of an irreducible set must be a multiple of the identity matrix. Thus we must have

$$T=\lambda I \tag{10}$$

for some constant λ. However, since T is the product of two matrices A and $\left(B^{\dagger}\right)^{-1}$, both of which have determinant 1, $\det T=1$ also, which means that $\lambda^2=1$ and $\lambda=\pm 1$. Therefore

$$\begin{aligned}
\left(B^{\dagger}\right)^{-1}A &= \pm I &(11)\\
A &= \pm B^{\dagger} &(12)
\end{aligned}$$

Thus the transformation 3 can be written as

$$\widehat{x}'=\pm A\widehat{x}A^{\dagger} \tag{13}$$

To eliminate the $-$ sign, suppose that

$$\widehat{x}' = -A\widehat{x}A^\dagger \tag{14}$$

A Lorentz transformation giving this result can be written as

$$\widehat{x}' = \widehat{\Lambda x} \tag{15}$$

where Λ is the 4×4 matrix giving the Lorentz transformation of the original 4-vector x. In the original 4-vector notation, we have

$$\begin{aligned} x'_\mu &= \sum_{\nu=0}^{3} \Lambda_{\mu\nu} x_\nu &(16)\\ &= (\Lambda x)_\mu &(17) \end{aligned}$$

From the relation betweenthe 4-vector and 2×2 matrix representations, we have

$$x'_\mu = \left\langle \sigma_\mu, \widehat{x}' \right\rangle \tag{18}$$

where $\langle \sigma_\mu, \widehat{x}' \rangle$ is the inner product of the two matrices. Therefore from 14

$$\begin{aligned} (\Lambda x)_\mu &= \left\langle \sigma_\mu, \widehat{x}' \right\rangle &(19)\\ &= -\left\langle \sigma_\mu, A\widehat{x}A^\dagger \right\rangle &(20)\\ &= -\left\langle \sigma_\mu, A\left(\sum_{\nu=0}^{3} \sigma_\nu x_\nu\right)A^\dagger \right\rangle &(21) \end{aligned}$$

If we choose $x = (1,0,0,0)$, we have

$$\begin{aligned} (\Lambda x)_0 &= \Lambda_{00} &(22)\\ &= -\left\langle \sigma_0, A\left(\sum_{\nu=0}^{3} \sigma_\nu x_\nu\right)A^\dagger \right\rangle &(23)\\ &= -\left\langle \sigma_0, A\sigma_0 A^\dagger \right\rangle &(24)\\ &= -\left\langle \sigma_0, AA^\dagger \right\rangle &(25)\\ &= -\frac{1}{2}\mathrm{Tr}\left(AA^\dagger\right) &(26)\\ &\le 0 &(27) \end{aligned}$$

where the penultimate line follows from the definition of the inner product. The last line follows because

$$\mathrm{Tr}\left(AA^{\dagger}\right)=|A_{11}|^{2}+|A_{22}|^{2}\geq 0 \tag{28}$$

Since we're requiring the transformation to be orthochronous, we must have $\Lambda_{00}\geq 1$, so we must exclude the $-$ sign in 13, giving 2.

Finally, we can show that the transformation matrix A is unique, up to a sign. We can prove this by supposing that there are two different $SL(2,\mathbb{C})$ matrices A and B that give the same transformation for all $\widehat{x}$, that is

$$A\widehat{x}A^{\dagger}=B\widehat{x}B^{\dagger} \tag{29}$$

This implies

$$B^{-1}A\widehat{x}A^{\dagger}\left(B^{\dagger}\right)^{-1} = \widehat{x} \tag{30}$$

$$= B^{-1}A\widehat{x}\left(B^{-1}A\right)^{\dagger} \tag{31}$$

We can now choose $\widehat{x}=I$, which shows that

$$\left(B^{-1}A\right)^{\dagger}=\left(B^{-1}A\right)^{-1} \tag{32}$$

which means (by definition), $B^{-1}A$ is unitary, so for all $\widehat{x}$

$$\widehat{x}=B^{-1}A\widehat{x}\left(B^{-1}A\right)^{-1} \tag{33}$$

This means that $B^{-1}A$ commutes with $\widehat{x}$ for all $\widehat{x}$ (that's the only way we can cancel $B^{-1}A$ off the RHS). Using the same argument as above, we can choose $\widehat{x}$ to be two of the Pauli matrices, which form an irreducible set. Since $B^{-1}A$ commutes with both these matrices, it must be a multiple λ of the identity:

$$B^{-1}A = \lambda I \tag{34}$$

$$A = \lambda B \tag{35}$$

Since $\det A=\det B=1$ and for a 2×2 matrix $\det(\lambda B)=\lambda^{2}\det B$, we have $\lambda^{2}=1$, so $\lambda=\pm 1$. Therefore A is unique up to a sign.

In summary, what we've done in this post is show that a restricted Lorentz transformation Λ (that is, one where $\det\Lambda=+1$ and $\Lambda_{00}\geq 1$) can be represented by a matrix $A\in SL(2,\mathbb{C})$ where A is unique up to a sign.

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