LORENTZ TRANSFORMATION AS PRODUCT OF A PURE BOOST AND PURE ROTATION

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References: W. Greiner & J. Reinhardt, *Field Quantization*, Springer-Verlag (1996), Chapter 2, Section 2.4.

Arthur Jaffe, *Lorentz transformations, rotations and boosts*, online notes available (at time of writing, Sep 2016) here.

Continuing our examination of general Lorentz transformations, we can now complete the demonstration that a general Lorentz transformation is the product of a pure boost (motion at a constant velocity) multiplied by a pure rotation. We'll follow Corollary IV.2 in Jaffe's article.

In the last post, we saw that we could write a general Lorentz transformation in the form

$$\widehat{\Lambda x} = A\widehat{x}A^{\dagger} \tag{1}$$

where x is the 4-vector of a spacetime event, Λ is the Lorentz transformation as a 4×4 matrix, A is a 2×2 matrix with complex elements and a hat over a symbol means we're looking at the 2×2 complex matrix representing that object. We also saw in the last post that this representation restricts

Jaffe goes through a rather involved proof that the transformation $\Lambda(A)$ defined by 1 is a member of the physically relevant group with $\det\Lambda=+1$ and $\Lambda_{00}\geq1$, but this involves a lot of somewhat obscure matrix theorems that I don't want to get into here, and these techniques don't seem to be required for the rest of the demonstration, so we'll just accept this fact for now.

What we really want to do is find out how we can calculate Λ given the 2×2 matrix A. We can do this by using the result we got earlier for the components of the 4-vector x:

$$\widehat{x}=\sum_{\mu=0}^{3}x_{\mu}\sigma_{\mu} \tag{2}$$

where the σ_{μ} are four Hermitian matrices:

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \tag{3}$$

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{4}$$

$$\sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{5}$$

$$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{6}$$

We can invert 2 to get

$$x_\nu = \langle \sigma_\nu, \widehat{x} \rangle = \frac{1}{2} \mathrm{Tr} \left(\sigma_\nu \widehat{x} \right) \tag{7}$$

Reverting back to the 4×4 matrix Λ (no hat), we have

$$x'_\mu = \sum_{\nu=0}^{3} \Lambda(A)_{\mu\nu} x_\nu \tag{8}$$

$$= (\Lambda(A) x)_\mu \tag{9}$$

$$= \left\langle \sigma_\mu, \widehat{\Lambda(A) x} \right\rangle \tag{10}$$

$$= \left\langle \sigma_\mu, A \widehat{x} A^\dagger \right\rangle \tag{11}$$

$$= \left\langle \sigma_\mu, A \sum_{\nu=0}^{3} x_\nu \sigma_\nu A^\dagger \right\rangle \tag{12}$$

$$= \sum_{\nu=0}^{3} \left\langle \sigma_\mu, A \sigma_\nu A^\dagger \right\rangle x_\nu \tag{13}$$

We used 1 in the fourth line and 2 in the fifth line. Comparing the first and last lines, we see that

$$\Lambda(A)_{\mu\nu} = \left\langle \sigma_\mu, A \sigma_\nu A^\dagger \right\rangle \tag{14}$$

$$= \frac{1}{2} \mathrm{Tr} \left(\sigma_\mu^\dagger A \sigma_\nu A^\dagger \right) \tag{15}$$

$$= \frac{1}{2} \mathrm{Tr} \left(\sigma_\mu A \sigma_\nu A^\dagger \right) \tag{16}$$

where in the last line we used the fact that all the σ_μ are Hermitian so that $\sigma_\mu^\dagger = \sigma_\mu$.

In order for $\Lambda(A)$ to be a valid Lorentz transformation, clearly its elements must be real numbers. We can show this is true as follows. The complex conjugate is represented by drawing a bar over a quantity. We get

$$\overline{\Lambda(A)_{\mu\nu}} = \frac{1}{2}\text{Tr}\left(\overline{\sigma_\mu A\sigma_\nu A^\dagger}\right) \tag{17}$$

We can now use the fact that the trace of a product of matrices remains unchanged if we cyclically permute the order of multiplication. In particular $\text{Tr}\left(XB^\dagger\right) = \text{Tr}\left(B^\dagger X\right)$. Also, $\text{Tr}\left(B^\dagger X\right) = \text{Tr}\left(\left(\overline{X^\dagger B}\right)^T\right) = \text{Tr}\left(\overline{X^\dagger B}\right)$ since the trace of a matrix is equal to the trace of its transpose. In 17, we can set $X^\dagger = \sigma_\mu$ and $B = A\sigma_\nu A^\dagger$ and use the fact that the σ_μ are all Hermitian so that $\sigma_\mu^\dagger = \sigma_\mu$:

$$\overline{\Lambda(A)_{\mu\nu}} = \frac{1}{2}\text{Tr}\left(\overline{\sigma_\mu A\sigma_\nu A^\dagger}\right) = \frac{1}{2}\text{Tr}\left(\left(A\sigma_\nu A^\dagger\right)^\dagger\sigma_\mu\right) \tag{18}$$

$$= \frac{1}{2}\text{Tr}\left(A\sigma_\nu A^\dagger\sigma_\mu\right) \tag{19}$$

$$= \frac{1}{2}\text{Tr}\left(\sigma_\mu A\sigma_\nu A^\dagger\right) \tag{20}$$

$$= \Lambda(A)_{\mu\nu} \tag{21}$$

where in the third line we cyclically permuted the matrices in the trace. Thus the elements of $\Lambda(A)$ are real.

Now we consider two cases. First, suppose that $A = U$, where U is a unitary matrix, so that $U^\dagger = U^{-1}$. From 16 we find that $\Lambda(U)_{00}$ is, using $\sigma_0 = I$:

$$\Lambda(U)_{00} = \frac{1}{2}\text{Tr}\left(\sigma_0 U\sigma_0 U^\dagger\right) \tag{22}$$

$$= \frac{1}{2}\text{Tr}\left(UU^\dagger\right) \tag{23}$$

$$= \frac{1}{2}\text{Tr}I \tag{24}$$

$$= 1 \tag{25}$$

The other elements in the first row and first column of Λ are all zero, as we can see by using 16 again:

$$\begin{aligned}
\Lambda(U)_{0i} &= \frac{1}{2}\mathrm{Tr}\left(\sigma_0 U \sigma_i U^\dagger\right) && (26)\\
&= \frac{1}{2}\mathrm{Tr}\left(U \sigma_i U^\dagger\right) && (27)\\
&= \frac{1}{2}\mathrm{Tr}\left(U^\dagger U \sigma_i\right) && (28)\\
&= \frac{1}{2}\mathrm{Tr}\left(\sigma_i\right) && (29)\\
&= 0 && (30)
\end{aligned}$$

since $\mathrm{Tr}\sigma_i = 0$ for $i = 1,2,3$. A similar argument works for the first column of $\Lambda(U)$ as well:

$$\begin{aligned}
\Lambda(U)_{i0} &= \frac{1}{2}\mathrm{Tr}\left(\sigma_i U \sigma_0 U^\dagger\right) && (31)\\
&= \frac{1}{2}\mathrm{Tr}\left(\sigma_i U U^\dagger\right) && (32)\\
&= \frac{1}{2}\mathrm{Tr}\left(\sigma_i\right) && (33)\\
&= 0 && (34)
\end{aligned}$$

For the other elements, we have

$$\begin{aligned}
\Lambda(U)_{ij} &= \frac{1}{2}\mathrm{Tr}\left(\sigma_i U \sigma_j U^\dagger\right) && (35)\\
&= \frac{1}{2}\mathrm{Tr}\left(\sigma_i \left(U^{-1}\right)^\dagger \sigma_j U^{-1}\right) && (36)\\
&= \frac{1}{2}\mathrm{Tr}\left(\sigma_j U^{-1} \sigma_i \left(U^{-1}\right)^\dagger\right) && (37)\\
&= \Lambda\left(U^{-1}\right)_{ji} && (38)\\
&= \left[\Lambda(U)\right]^{-1}_{ji} && (39)
\end{aligned}$$

That is

$$\left[\Lambda(U)\right]^T = \Lambda(U)^{-1} \tag{40}$$

so that

$$\Lambda = \left[\begin{array}{cc} 1 & 0 \\ 0 & \mathcal{R} \end{array}\right] \tag{41}$$

where $\mathcal{R}$ is a 3×3 matrix, and the 0s represent 3 zero components in the top row and first column. In other words, when $A = U$, Λ is a pure rotation.

The other case we need to examine is when $A=H$, where H is a Hermitian matrix, so that $H^{\dagger}=H$. In that case, from 16

$$
\begin{aligned}
\Lambda(H)_{\mu\nu} &= \frac{1}{2}\mathrm{Tr}\left(\sigma_{\mu}H\sigma_{\nu}H\right) && (42)\\
&= \frac{1}{2}\mathrm{Tr}\left(H\sigma_{\mu}H\sigma_{\nu}\right) && (43)\\
&= \frac{1}{2}\mathrm{Tr}\left(\sigma_{\nu}H\sigma_{\mu}H\right) && (44)\\
&= \Lambda(H)_{\nu\mu} && (45)
\end{aligned}
$$

so $\Lambda(H)$ is a symmetric matrix. (We used two cyclic permutations in the trace here.) Although we haven't proved that a symmetric Lorentz transformation always represents a pure boost, this has been verified (see, for example, Wikipedia; I can't be bothered going through it all here).

Now we are ready to get our final result. To do this, we need to use a theorem from matrix algebra which says that every matrix A in the group $SL(2,\mathbb{C})$ (that is, a 2×2 matrix with complex elements and determinant +1) has a unique polar decomposition into a strictly positive Hermitian matrix H and a unitary matrix U, so that we always have

$$
A=HU \tag{46}
$$

To connect this with what we've done above, we can define

$$
\begin{aligned}
H &= \left(AA^{\dagger}\right)^{1/2} && (47)\\
U &= H^{-1}A=\left(AA^{\dagger}\right)^{1/2}A && (48)
\end{aligned}
$$

[The square root of a matrix is defined to be the matrix $S=A^{1/2}$ so that $S^{2}=A$.] This definition is consistent with H being Hermitian, since

$$
\begin{aligned}
\left(S^{2}\right)^{\dagger} &= A^{\dagger}=A && (49)\\
&= (SS)^{\dagger} && (50)\\
&= \left(S^{\dagger}\right)^{2} && (51)\\
&= S^{2} && (52)
\end{aligned}
$$

Thus if we restrict S to be the positive square root, we must have $S^{\dagger}=S$. The definition is also consistent with U being unitary, since

$$\begin{aligned} UU^{\dagger} &= \left(H^{-1}A\right)\left(H^{-1}A\right)^{\dagger} && (53)\\ &= H^{-1}AA^{\dagger}H^{-1} && (54)\\ &= \left(AA^{\dagger}\right)^{-1/2}AA^{\dagger}\left(AA^{\dagger}\right)^{-1/2} && (55)\\ &= I && (56)\end{aligned}$$

[We define $\left(AA^{\dagger}\right)^{-1/2}$ to be the inverse of $\left(AA^{\dagger}\right)^{1/2}$.]

Therefore, we can uniquely decompose any Lorentz transformation $\Lambda(A)$ into

$$\Lambda(A) = \Lambda(H)\Lambda(U) \tag{57}$$

that is, the product of a pure rotation and a pure boost.

PINGBACKS

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