

## NOETHER'S THEOREM AND CONSERVATION OF ANGULAR MOMENTUM

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References: W. Greiner & J. Reinhardt, *Field Quantization*, Springer-Verlag (1996), Chapter 2, Section 2.4.

Now that we've seen that a general Lorentz transformation can be represented as a product of a pure boost and a pure 3-d rotation, we can return to Noether's theorem and see what conserved property it predicts when we require a physical system to be invariant under a Lorentz transformation. As usual, we consider an infinitesimal transformation, which we can write as

$$(1) \quad x'^{\mu} = x^{\mu} + \delta\omega^{\mu\nu}x_{\nu}$$

where  $\delta\omega^{\mu\nu}$  is the infinitesimal rotation in 4-dimensional spacetime. Here, we are treating a pure boost as a rotation; for example, a boost in the  $x_1$  direction is given by the Lorentz transformation

$$(2) \quad x' = \Lambda x$$

where

$$(3) \quad \Lambda = \begin{bmatrix} \cosh\chi & \sinh\chi & 0 & 0 \\ \sinh\chi & \cosh\chi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

for some 'angle'  $\chi$ . This reduces to the more familiar form found in introductory relativity courses if we set

$$(4) \quad \cosh\chi \equiv \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$(5) \quad \sinh\chi \equiv \beta\gamma = \frac{\beta}{\sqrt{1-\beta^2}}$$

Returning to 1, we require, to first order in  $\delta\omega_{\mu\nu}$ , that the Minkowski length of the 4-vector is the same before and after the transformation. That is,

$$\begin{aligned}
 (6) \quad x'^{\mu}x'_{\mu} &= (x^{\mu} + \delta\omega^{\mu\sigma}x_{\sigma}) (x_{\mu} + \delta\omega_{\mu}^{\tau}x_{\tau}) \\
 (7) &= x^{\mu}x_{\mu} + \delta\omega^{\mu\sigma}x_{\sigma}x_{\mu} + \delta\omega_{\mu}^{\tau}x_{\tau}x^{\mu} \\
 (8) &= x^{\mu}x_{\mu} + \delta\omega^{\mu\sigma}x_{\sigma}x_{\mu} + \delta\omega^{\mu\tau}x_{\tau}x_{\mu} \\
 (9) &= x^{\mu}x_{\mu} + 2\delta\omega^{\mu\nu}x_{\mu}x_{\nu}
 \end{aligned}$$

In the last line, we renamed the dummy indices  $\sigma$  and  $\tau$  to  $\nu$ . To first order, we require the last term in the last line to be zero for all  $x_{\mu}$ , which means we must impose a condition on  $\delta\omega^{\mu\nu}$ . We can write this term as

$$(10) \quad 2\delta\omega^{\mu\nu}x_{\mu}x_{\nu} = x_{\mu}x_{\nu}(\delta\omega^{\mu\nu} + \delta\omega^{\nu\mu})$$

From this, we see that we must have

$$(11) \quad \delta\omega^{\mu\nu} = -\delta\omega^{\nu\mu}$$

so  $\delta\omega^{\mu\nu}$  must be antisymmetric.

Incidentally, if this condition seems to be violated in the pure boost matrix 3, remember that 2 is an ordinary matrix product, while the last term in 1 is the product of a tensor and 4-vector, and thus includes the effect of the metric tensor

$$(12) \quad g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

To first order in  $\chi$ , 3 is

$$(13) \quad \Lambda = \begin{bmatrix} 1 & \chi & 0 & 0 \\ \chi & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

while for the infinitesimal rotation, we have

$$(14) \quad \delta\omega = \begin{bmatrix} 0 & -\chi & 0 & 0 \\ \chi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In matrix notation, 1 becomes

$$(15) \quad x' = \delta\omega \times g \times x$$

from which we can see that this gives the same result as 2.

In order to apply Noether's theorem, we also need to know how the fields transform under a Lorentz transformation. The assumption is that, for infinitesimal transformations, the transformed field  $\phi'_r(x')$  depends linearly on both the original fields  $\phi_r(x)$  and on the rotation  $\delta\omega_{\mu\nu}$ . That is, we assume that

$$(16) \quad \phi'_r(x') = \phi_r(x) + \frac{1}{2}\delta\omega_{\mu\nu}(I^{\mu\nu})_{rs}\phi_s(x)$$

where  $I^{\mu\nu}$  are the infinitesimal generators of the Lorentz transformation. G & R don't really explain this, apart from giving a reference to another book, but we won't need to delve into the details to get the result needed in this post, so we'll leave it for now.

From here, it's a matter of plugging 1 and 16 into the equations for Noether's theorem and seeing what comes out. Noether's theorem says that

$$(17) \quad \partial^\mu f_\mu(x) = 0$$

where

$$(18) \quad f_\mu(x) \equiv \frac{\partial\mathcal{L}(x)}{\partial(\partial^\mu\phi_r)}(\delta\phi_r(x) - \partial^\nu\phi_r\delta x_\nu) + \mathcal{L}(x)\delta x_\mu$$

$$(19) \quad = \frac{\partial\mathcal{L}(x)}{\partial(\partial^\mu\phi_r)}\delta\phi_r(x) - \left(\frac{\partial\mathcal{L}(x)}{\partial(\partial^\mu\phi_r)}\partial_\nu\phi_r - g_{\mu\nu}\mathcal{L}(x)\right)\delta x^\nu$$

From 16

$$(20) \quad \delta\phi_r(x) = \phi'_r(x') - \phi_r(x)$$

$$(21) \quad = \frac{1}{2}\delta\omega_{\mu\nu}(I^{\mu\nu})_{rs}\phi_s(x)$$

$$(22) \quad = \frac{1}{2}\delta\omega_{\nu\lambda}(I^{\nu\lambda})_{rs}\phi_s(x)$$

In the last line, we renamed the indices  $\mu$  and  $\nu$  to  $\nu$  and  $\lambda$  respectively to avoid confusing the  $\mu$  in the first term of 19 with any of the indices in  $\delta\phi_r(x)$ .

and from 1

$$(23) \quad \delta x^\nu = x'^\nu - x^\nu$$

$$(24) \quad = \delta\omega^{\nu\lambda} x_\lambda$$

We also had the energy-momentum tensor

$$(25) \quad T_{\mu\nu} \equiv \frac{\partial \mathcal{L}(x)}{\partial (\partial^\mu \phi_r)} \partial_\nu \phi_r - g_{\mu\nu} \mathcal{L}(x)$$

Putting all this together, we have

$$(26) \quad f_\mu(x) = \frac{\partial \mathcal{L}(x)}{\partial (\partial^\mu \phi_r)} \frac{1}{2} \delta\omega_{\nu\lambda} (I^{\nu\lambda})_{rs} \phi_s(x) - T_{\mu\nu} \delta\omega^{\nu\lambda} x_\lambda$$

Because  $\delta\omega^{\nu\lambda} = -\delta\omega^{\lambda\nu}$

$$(27) \quad T_{\mu\nu} \delta\omega^{\nu\lambda} x_\lambda = \frac{1}{2} [T_{\mu\nu} \delta\omega^{\nu\lambda} x_\lambda - T_{\mu\nu} \delta\omega^{\lambda\nu} x_\lambda]$$

$$(28) \quad = \frac{1}{2} [T_{\mu\nu} \delta\omega^{\nu\lambda} x_\lambda - T_{\mu\lambda} \delta\omega^{\nu\lambda} x_\nu]$$

$$(29) \quad = \frac{1}{2} \delta\omega^{\nu\lambda} [T_{\mu\nu} x_\lambda - T_{\mu\lambda} x_\nu]$$

In the second line, we swapped the dummy indices  $\lambda$  and  $\nu$  in the second term, which is allowed because both indices are summed. Therefore

$$(30) \quad f_\mu(x) = \frac{\partial \mathcal{L}(x)}{\partial (\partial^\mu \phi_r)} \frac{1}{2} \delta\omega_{\nu\lambda} (I^{\nu\lambda})_{rs} \phi_s(x) - \frac{1}{2} \delta\omega^{\nu\lambda} [T_{\mu\nu} x_\lambda - T_{\mu\lambda} x_\nu]$$

$$(31) \quad = \frac{\partial \mathcal{L}(x)}{\partial (\partial^\mu \phi_r)} \frac{1}{2} \delta\omega^{\nu\lambda} (I_{\nu\lambda})_{rs} \phi_s(x) - \frac{1}{2} \delta\omega^{\nu\lambda} [T_{\mu\nu} x_\lambda - T_{\mu\lambda} x_\nu]$$

$$(32) \quad = \frac{1}{2} \delta\omega^{\nu\lambda} \left[ \frac{\partial \mathcal{L}(x)}{\partial (\partial^\mu \phi_r)} (I_{\nu\lambda})_{rs} \phi_s(x) - (T_{\mu\nu} x_\lambda - T_{\mu\lambda} x_\nu) \right]$$

$$(33) \quad \equiv \frac{1}{2} \delta\omega^{\nu\lambda} M_{\mu\nu\lambda}(x)$$

In the second line, we swapped the positions of the  $\nu\lambda$  indices on the terms  $\delta\omega_{\nu\lambda} (I^{\nu\lambda})_{rs}$ . This is OK provided they are both summed over. The last line defines the term  $M_{\mu\nu\lambda}(x)$ .

Because the infinitesimal rotations are arbitrary (subject to the condition that  $\delta\omega^{\nu\lambda}$  is antisymmetric), we can choose all of them to be zero except for one. For each such choice, we have a different  $f_\mu(x)$ , which leads to a conservation law for each choice. From Noether's theorem, the quantity that is conserved is the integral of  $f_0$  over 3-space, so we have the conserved quantities

$$(34) \quad M_{\nu\lambda} = \int d^3x \left[ \frac{\partial \mathcal{L}(x)}{\partial (\partial^0 \phi_r)} (I_{\nu\lambda})_{rs} \dot{\phi}_s(x) - (T_{0\nu}x_\lambda - T_{0\lambda}x_\nu) \right]$$

$$(35) \quad = \int d^3x \left[ T_{0\lambda}x_\nu - T_{0\nu}x_\lambda + \frac{\partial \mathcal{L}(x)}{\partial (\partial^0 \phi_r)} (I_{\nu\lambda})_{rs} \dot{\phi}_s(x) \right]$$

From 25, we see that the first two terms are

$$(36) \quad T_{0\lambda}x_\nu - T_{0\nu}x_\lambda = x_\nu \frac{\partial \mathcal{L}}{\partial \dot{\phi}_r} \partial_\lambda \phi_r - x_\lambda \frac{\partial \mathcal{L}}{\partial \dot{\phi}_r} \partial_\nu \phi_r$$

$$(37) \quad = x_\nu \pi_r \partial_\lambda \phi_r - x_\lambda \pi_r \partial_\nu \phi_r$$

where  $\pi_r$  is the conjugate momentum density, defined by

$$(38) \quad \pi_r \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}_r}$$

Using the physical momentum density

$$(39) \quad p_\lambda = \pi_r \partial_\lambda \phi_r$$

we find that

$$(40) \quad T_{0\lambda}x_\nu - T_{0\nu}x_\lambda = x_\nu p_\lambda - x_\lambda p_\nu$$

This is one component of the angular momentum density  $\mathbf{r} \times \mathbf{p}$ , so the integral

$$(41) \quad \int d^3x (T_{0\lambda}x_\nu - T_{0\nu}x_\lambda)$$

is one component of the total angular momentum.

The other term in 35 is

$$(42) \quad \int d^3x \frac{\partial \mathcal{L}(x)}{\partial (\partial^0 \phi_r)} (I_{\nu\lambda})_{rs} \dot{\phi}_s(x)$$

depends on the generators  $I_{\nu\lambda}$ , and thus on the specific way in which the fields transform. G & R tell us that this term describes the spin angular momentum, but at this stage, we just have to accept this on faith.

In any case, the overall conservation rule 35 shows that the sum of the 'traditional' angular momentum from the first two terms in the integrand, together with this mysterious other term, is a conserved quantity, so interpreting it as some other form of angular momentum seems reasonable. We'll just have to wait and see how this plays out.