NONRELATIVISTIC FIELD THEORY - NUMBER, CREATION AND ANNIHILATION OPERATORS

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: W. Greiner & J. Reinhardt, *Field Quantization*, Springer-Verlag (1996), Chapter 3, Section 3.2.

Post date: 29 Jan 2018.

For a time-independent potential, we've seen that the Hamiltonian in nonrelativistic field theory can be written as

$$\hat{H}=\sum_{i}\hat{a}_{i}^{\dagger}\left(t\right)\hat{a}_{i}\varepsilon_{i} \tag{1}$$

whereε_{i} is the energy of stationary state i and the $\hat{a}_{i}\left(t\right)$ are the coefficients in a Fourier expansion of the field operator

$$\hat{\psi}\left(\mathbf{x},t\right)=\sum_{i}\hat{a}_{i}\left(t\right)u_{i}\left(\mathbf{x}\right) \tag{2}$$

$$\hat{\psi}^{\dagger}\left(\mathbf{x},t\right)=\sum_{i}\hat{a}_{i}^{\dagger}\left(t\right)u_{i}^{*}\left(\mathbf{x}\right) \tag{3}$$

These operators and their adjoints satisfy the commutation relations

$$\left[\hat{a}_{i}\left(t\right),\hat{a}_{j}^{\dagger}\left(t\right)\right]=\delta_{ij} \tag{4}$$

$$\left[\hat{a}_{i}\left(t\right),\hat{a}_{j}\left(t\right)\right]=\left[\hat{a}_{i}^{\dagger}\left(t\right),\hat{a}_{j}^{\dagger}\left(t\right)\right]=0 \tag{5}$$

Equation 1 is interpreted as the energy of a collection of particles where we have a number n_{i} of each eigenstate where

$$n_{i}\left(t\right)\equiv\hat{a}_{i}^{\dagger}\left(t\right)\hat{a}_{i}\left(t\right) \tag{6}$$

The total number of particles is then

$$\hat{N}=\sum_{i}\hat{n}_{i}=\sum_{i}\hat{a}_{i}^{\dagger}\hat{a}_{i} \tag{7}$$

Because of the commutation relations we can see that

$$\left[\hat{N},\hat{H}\right]=\left[\sum_{i}\hat{a}_{i}^{\dagger}\hat{a}_{i},\sum_{j}\hat{a}_{j}^{\dagger}\hat{a}_{j}\varepsilon_{i}\right]=0 \tag{8}$$

since terms with $i \neq j$ all commute, and terms with $i = j$ commute since $\hat{a}_i^\dagger \hat{a}_i$ always commutes with itself. Since the time derivative of an operator is given by

$$\dot{\hat{N}} = \frac{1}{i\hbar}\left[\hat{N}, \hat{H}\right] \tag{9}$$

in this case, we see that the total particle number is a constant in time.

If two hermitian operators commute, there is a basis in which both operators are diagonal or, in other words, we can find a basis of eigenstates that are common to both operators. Since both $\hat{N}$ and $\hat{H}$ are written in terms of the particle number operators $\hat{n}_i$, such a basis is one in which we specify the numbers n_i of particles in all the states. That is, the state $|n_1, n_2, \ldots, n_i, \ldots\rangle$ is an eigenstate of $\hat{n}_i$ with eigenvalue n_i:

$$\hat{n}_i |n_1, n_2, \ldots, n_i, \ldots\rangle = n_i |n_1, n_2, \ldots, n_i, \ldots\rangle \tag{10}$$

and this state is also an eigenstate of the total number operator $\hat{N}$:

$$\hat{N} |n_1, n_2, \ldots, n_i, \ldots\rangle = \left(\sum_i \hat{n}_i\right) |n_1, n_2, \ldots, n_i, \ldots\rangle \tag{11}$$

$$= \left(\sum_i n_i\right) |n_1, n_2, \ldots, n_i, \ldots\rangle \tag{12}$$

$$= n |n_1, n_2, \ldots, n_i, \ldots\rangle \tag{13}$$

where the total number of particles is

$$n \equiv \sum_i n_i \tag{14}$$

As usual, the states in this basis (sometimes called Fock space) are orthonormal, so that

$$\left\langle n_1', n_2', \ldots | n_1, n_2 \right\rangle = \delta_{n_1' n_1} \delta_{n_2' n_2} \cdots \tag{15}$$

As Greiner shows in his derivation of eqn 3.31, we can use the commutator 4 to show what happens when we apply $\hat{a}_i^\dagger$ to a state $|n_1, n_2, \ldots, n_i, \ldots\rangle$ to get

$$\hat{N}\hat{a}_i^\dagger |n_1, n_2, \ldots, n_i, \ldots\rangle = (n+1)\, \hat{a}_i^\dagger |n_1, n_2, \ldots, n_i, \ldots\rangle \tag{16}$$

A similar argument works for $\hat{a}_i$:

$$\hat{N}a_i|n_1,n_2,\ldots,n_i,\ldots\rangle = \sum_j \hat{a}_j^\dagger \hat{a}_j \hat{a}_i |n_1,n_2,\ldots,n_i,\ldots\rangle \tag{17}$$

$$= \sum_j \hat{a}_j^\dagger \hat{a}_i \hat{a}_j |n_1,n_2,\ldots,n_i,\ldots\rangle \tag{18}$$

$$= \sum_j \left(\hat{a}_i \hat{a}_j^\dagger - \delta_{ij}\right) \hat{a}_j |n_1,n_2,\ldots,n_i,\ldots\rangle \tag{19}$$

$$= \left[\sum_j \left(\hat{a}_i \hat{a}_j^\dagger \hat{a}_j\right) - \hat{a}_i\right] |n_1,n_2,\ldots,n_i,\ldots\rangle \tag{20}$$

$$= \left(\hat{N}-1\right) \hat{a}_i |n_1,n_2,\ldots,n_i,\ldots\rangle \tag{21}$$

$$= (n-1)\, \hat{a}_i |n_1,n_2,\ldots,n_i,\ldots\rangle \tag{22}$$

Thus $\hat{a}_i^\dagger$ is a creation operator, since its effect is to increase the number of type i particles by 1. Similarly, a_i is an annihilation operator, since it decreases the number of type i particles by 1. We can generate states corresponding to all possible combinations of particle numbers by starting with the vacuum state $|0\rangle$ which contains no particles and applying the appropriate creation operators successively. The vacuum state is defined to be a state which is destroyed by applying any annihilation operator

$$\hat{a}_i|0\rangle = 0 \tag{23}$$

Note that $|0\rangle$ is a vector in a vector space, and is *not* equal to the number 0, which is just an ordinary number, not a vector.

However, merely applying the creation operator $\hat{a}_i^\dagger$ to a state does not necessarily produce a new state that is properly normalized. Thus, in general, to produce the state $|n_1,n_2,\ldots,n_i,\ldots\rangle$ we need to apply creation operators $\left(\hat{a}_1^\dagger\right)^{n_1}$, $\left(\hat{a}_2^\dagger\right)^{n_2}$, $\left(\hat{a}_i^\dagger\right)^{n_i}$ and so on, and then multiply the result by a constant $C_{n_1,n_2,\ldots}$ to get a properly normalized state. Greiner shows in Exercise 3.1 how to derive this constant using mathematical induction. To do this, we 'guess' the solution and then prove it. The proposal is that

$$C_{n_1,n_2,\ldots} = \frac{1}{\sqrt{n_1!n_2!\ldots}} \tag{24}$$

We begin with the vacuum state, which is assumed to be normalized so that

$$\langle 0|0\rangle = 1 \tag{25}$$

To get a state where $n_i = 1$ for one particular i (with all other particle numbers being zero), we apply $\hat{a}_i^\dagger$ and find the square magnitude of the resulting vector. This is

$$\left\langle \hat{a}_i^\dagger 0 \middle| \hat{a}_i^\dagger 0 \right\rangle = \left\langle 0 \middle| \hat{a}_i \hat{a}_i^\dagger 0 \right\rangle \tag{26}$$

We can now apply the commutator 4 to get

$$\left\langle 0 \middle| \hat{a}_i \hat{a}_i^\dagger 0 \right\rangle = \left\langle 0 \middle| 1 + \hat{a}_i^\dagger \hat{a}_i 0 \right\rangle \tag{27}$$

$$= \langle 0|0\rangle + 0 \tag{28}$$

$$= 1 \tag{29}$$

The second line follows from 23. Thus 24 is true for a single particle. This constitutes the anchor step for the induction.

The inductive step starts by assuming that we have a state where

$$|n_1, n_2, \ldots, n_i, \ldots\rangle = \frac{1}{\sqrt{n_1! n_2! \ldots n_i! \ldots}} \left(\hat{a}_1^\dagger\right)^{n_1} \left(\hat{a}_2^\dagger\right)^{n_2} \ldots \left(\hat{a}_i^\dagger\right)^{n_i} \ldots |0\rangle \tag{30}$$

$$= C_{n_1, n_2, \ldots, n_i \ldots} \left(\hat{a}_1^\dagger\right)^{n_1} \left(\hat{a}_2^\dagger\right)^{n_2} \ldots \left(\hat{a}_i^\dagger\right)^{n_i} \ldots |0\rangle \tag{31}$$

We then apply $\hat{a}_i^\dagger$ to this state and calculate the square magnitude of the resulting state. This gives us

$$\left\langle n_1, n_2, \ldots, n_i, \ldots \middle| \hat{a}_i \hat{a}_i^\dagger \middle| n_1, n_2, \ldots, n_i, \ldots \right\rangle = \tag{32}$$

$$\left|C_{n_1, n_2, \ldots, n_i+1 \ldots}\right|^2 \left\langle 0 \middle| (\hat{a}_1)^{n_1} (\hat{a}_2)^{n_2} \ldots (\hat{a}_i)^{n_i+1} \left(\hat{a}_1^\dagger\right)^{n_1} \left(\hat{a}_2^\dagger\right)^{n_2} \ldots \left(\hat{a}_i^\dagger\right)^{n_i+1} \middle| 0 \right\rangle \tag{33}$$

The extra $\hat{a}_i$ commutes with all the $\left(\hat{a}_j^\dagger\right)^{n_j}$ terms provided that $i \neq j$, so it propagates to the right until it encounters the $\left(\hat{a}_i^\dagger\right)^{n_i}$ term. At this point, we must apply the commutator 4 n_i times to move the $\hat{a}_i$ operator through this term. Each commutator contributes a 1 just as we picked up the 1 when doing the calculation 27 for a single particle. Once the $\hat{a}_i$ is passed through the $\left(\hat{a}_i^\dagger\right)^{n_i}$ term, it commutes with all remaining $\left(\hat{a}_j^\dagger\right)^{n_j}$ terms until it encounters the vacuum state, which it destroys. That is, the overall effect is

$$\hat{a}_i \left(\hat{a}_i^\dagger\right)^{n_i+1} \rightarrow \left(\hat{a}_i^\dagger\right)^{n_i} (n_i + 1) \tag{34}$$

The result is that the overall inner product picks up a factor of (n_i+1), so we have

$$\left\langle n_1,n_2,\ldots,n_i,\ldots\left|\hat{a}_i\hat{a}_i^\dagger\right|n_1,n_2,\ldots,n_i,\ldots\right\rangle = \tag{35}$$

$$(n_i+1)\left|C_{n_1,n_2,\ldots,n_i+1\ldots}\right|^2\left\langle 0\left|(\hat{a}_1)^{n_1}(\hat{a}_2)^{n_2}\ldots(\hat{a}_i)^{n_i}\left(\hat{a}_1^\dagger\right)^{n_1}\left(\hat{a}_2^\dagger\right)^{n_2}\ldots\left(\hat{a}_i^\dagger\right)^{n_i}\right|0\right\rangle = \tag{36}$$

$$(n_i+1)\frac{\left|C_{n_1,n_2,\ldots,n_i+1\ldots}\right|^2}{\left|C_{n_1,n_2,\ldots,n_i\ldots}\right|^2} = 1 \tag{37}$$

where the '= 1' in the last line is the required condition. Solving, we find

We could multiply $C_{n_1,n_2,\ldots,n_i+1\ldots}$ by some phase factor, but it's easiest to take the phase to be +1.

$$C_{n_1,n_2,\ldots,n_i+1\ldots} = \frac{1}{\sqrt{n_i+1}}C_{n_1,n_2,\ldots,n_i\ldots} \tag{38}$$

$$= \frac{1}{\sqrt{n_1!n_2!\ldots(n_i+1)!\ldots}} \tag{39}$$

which is what we wanted to prove.

Finally, using an argument similar to that for the harmonic oscillator, we can work out the effect of the creation and annihilation operators. We have from 6 and 10

$$\left\langle n_1,n_2,\ldots,n_i,\ldots\left|\hat{a}_i^\dagger\hat{a}_i\right|n_1,n_2,\ldots,n_i,\ldots\right\rangle = n_i\left\langle n_1,n_2,\ldots,n_i,\ldots|n_1,n_2,\ldots,n_i,\ldots\right\rangle = n_i \tag{40}$$

However, we can also write this as

$$\left\langle \hat{a}_i n_1,n_2,\ldots,n_i,\ldots|\hat{a}_i n_1,n_2,\ldots,n_i,\ldots\right\rangle = n_i \tag{41}$$

Since $\hat{a}_i$ reduces the particle number n_i by 1, we must have

$$\hat{a}_i\left|n_1,n_2,\ldots,n_i,\ldots\right\rangle = A\left|n_1,n_2,\ldots,n_i-1,\ldots\right\rangle \tag{42}$$

where the constant A must be, in order to satisfy 41

$$A = \sqrt{n_i} \tag{43}$$

and

$$\hat{a}_i\left|n_1,n_2,\ldots,n_i,\ldots\right\rangle = \sqrt{n_i}\left|n_1,n_2,\ldots,n_i-1,\ldots\right\rangle \tag{44}$$

Similarly, we have

$$\left\langle n_1,n_2,\ldots,n_i,\ldots\left|\hat{a}_i\hat{a}_i^{\dagger}\right|n_1,n_2,\ldots,n_i,\ldots\right\rangle = \left\langle n_1,n_2,\ldots,n_i,\ldots\left|1+\hat{a}_i^{\dagger}\hat{a}_i\right|n_1,n_2,\ldots,n_i,\ldots\right\rangle \tag{45}$$

$$= (1+n_i)\left\langle n_1,n_2,\ldots,n_i,\ldots|n_1,n_2,\ldots,n_i,\ldots\right\rangle \tag{46}$$

$$= 1+n_i \tag{47}$$

This gives

$$\hat{a}_i^{\dagger}\left|n_1,n_2,\ldots,n_i,\ldots\right\rangle = \sqrt{n_i+1}\left|n_1,n_2,\ldots,n_i+1,\ldots\right\rangle \tag{48}$$

As a result, the matrix elements of $\hat{a}_i^{\dagger}$ and $\hat{a}_i$ in the $\left|n_1,n_2,\ldots,n_i,\ldots\right\rangle$ basis are

$$\left\langle n_1',n_2',\ldots,n_i',\ldots\left|\hat{a}_i^{\dagger}\right|n_1,n_2,\ldots,n_i,\ldots\right\rangle = \sqrt{n_i+1}\,\delta_{n_1'n_1}\delta_{n_2'n_1}\ldots\delta_{n_i'(n_i+1)}\cdots \tag{49}$$

$$\left\langle n_1',n_2',\ldots,n_i',\ldots\left|\hat{a}_i\right|n_1,n_2,\ldots,n_i,\ldots\right\rangle = \sqrt{n_i}\,\delta_{n_1'n_1}\delta_{n_2'n_1}\ldots\delta_{n_i'(n_i-1)}\cdots \tag{50}$$

PINGBACKS

Pingback: Nonrelativistic field theory - Fock space to position space