NONRELATIVISTIC FIELD THEORY - FOCK SPACE TO POSITION SPACE

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References: W. Greiner & J. Reinhardt, *Field Quantization*, Springer-Verlag (1996), Chapter 3, Section 3.2.

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In nonrelativistic field theory, we've seen that one basis of the multi-particle space can be given by the vectors

$$|n_1,n_2,\ldots,n_i,\ldots\rangle \tag{1}$$

where each n_i represents a number of particles in energy state ε_i. This is known as Fock space. We'd like to see the relation of Fock space to the more usual position space, in which the particles are represented by their positions in space. Greiner therefore defines the states

$$|\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n;t\rangle \equiv \frac{1}{\sqrt{n!}}\hat{\psi}^{\dagger}(\mathbf{x}_1,t)\,\hat{\psi}^{\dagger}(\mathbf{x}_2,t)\ldots\hat{\psi}^{\dagger}(\mathbf{x}_n,t)\,|0\rangle \tag{2}$$

where the field operators are defined as Fourier series

$$\hat{\psi}(\mathbf{x},t) = \sum_i \hat{a}_i(t)\,u_i(\mathbf{x}) \tag{3}$$

$$\hat{\psi}^{\dagger}(\mathbf{x},t) = \sum_i \hat{a}_i^{\dagger}(t)\,u_i^*(\mathbf{x}) \tag{4}$$

and satisfy the commutation relations

$$\left[\hat{\psi}(\mathbf{x},t),\hat{\psi}^{\dagger}\left(\mathbf{x}',t\right)\right] = \delta^3\left(\mathbf{x}-\mathbf{x}'\right) \tag{5}$$

with all other commutators being zero.

The definition 2 represents a collection of n particles localized in space at the points $\mathbf{x}_i$ for $i=1,\ldots,n$. We can now define the number operator as

$$\hat{N}_V \equiv \int_V d^3x\;\hat{\psi}^{\dagger}(\mathbf{x},t)\,\hat{\psi}(\mathbf{x},t) \tag{6}$$

where the integral is over some finite volume V. We can use the commutators 3 and 4 to work out the commutator of $\hat{N}_V$ (we'll suppress the time dependence from here on, as it's always the same):

$$\left[\hat{N}_V, \hat{\psi}^\dagger(\mathbf{x})\right] = \int_V d^3x' \left[\hat{\psi}^\dagger(\mathbf{x}), \hat{\psi}^\dagger(\mathbf{x}')\hat{\psi}(\mathbf{x}')\right] \tag{7}$$

$$= \int_V d^3x' \, \hat{\psi}^\dagger(\mathbf{x}') \left[\hat{\psi}^\dagger(\mathbf{x}), \hat{\psi}(\mathbf{x}')\right] \tag{8}$$

$$= \int_V d^3x' \, \hat{\psi}^\dagger(\mathbf{x}') \, \delta^3(\mathbf{x}-\mathbf{x}') \tag{9}$$

$$= \begin{cases} \hat{\psi}^\dagger(\mathbf{x}) & \text{if } \mathbf{x} \in V \\ 0 & \text{if } \mathbf{x} \notin V \end{cases} \tag{10}$$

Using this commutator, we can work out the effect of operating on 2 with $\hat{N}_V$. Consider first the two-particle case, where we have

$$\hat{N}_V |\mathbf{x}_1, \mathbf{x}_2\rangle \equiv \frac{1}{\sqrt{2!}} \hat{N}_V \hat{\psi}^\dagger(\mathbf{x}_1) \hat{\psi}^\dagger(\mathbf{x}_2) |0\rangle \tag{11}$$

Suppose that both $\mathbf{x}_1$ and $\mathbf{x}_2$ lie within V. Then we have

$$\hat{N}_V \hat{\psi}^\dagger(\mathbf{x}_1) \hat{\psi}^\dagger(\mathbf{x}_2) |0\rangle = \left(\hat{\psi}^\dagger(\mathbf{x}_1) + \hat{\psi}^\dagger(\mathbf{x}_1) \hat{N}_V\right) \hat{\psi}^\dagger(\mathbf{x}_2) |0\rangle \tag{12}$$

$$= \left(\hat{\psi}^\dagger(\mathbf{x}_1) \hat{\psi}^\dagger(\mathbf{x}_2) + \hat{\psi}^\dagger(\mathbf{x}_1) \hat{\psi}^\dagger(\mathbf{x}_2) + \hat{\psi}^\dagger(\mathbf{x}_1) \hat{\psi}^\dagger(\mathbf{x}_2) \hat{N}_V\right) |0\rangle \tag{13}$$

$$= 2\hat{\psi}^\dagger(\mathbf{x}_1) \hat{\psi}^\dagger(\mathbf{x}_2) |0\rangle + 0 \tag{14}$$

$$= 2\hat{\psi}^\dagger(\mathbf{x}_1) \hat{\psi}^\dagger(\mathbf{x}_2) |0\rangle \tag{15}$$

In the third line we've used the fact that $\hat{N}_V |0\rangle = 0$, since the first operator in 6 to operate on the vacuum state $|0\rangle$ is the annihilation operator $\hat{\psi}(\mathbf{x})$.

Using a similar argument, we can see that any particles lying outside V will not be counted since for those particles $\left[\hat{N}_V, \hat{\psi}^\dagger(\mathbf{x})\right] = 0$ and the number operator just passes through the corresponding creation operator. The generalization of 15 shows that in general

$$\hat{N}_V |\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n; t\rangle = n_V |\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n; t\rangle \tag{16}$$

where n_V is the number of particles within V. Thus the states $|\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n; t\rangle$ are eigenstates of $\hat{N}_V$.

Finally, we can work out the normalization of the position space states, again by applying the commutators. Again, we can start with a 2-particle state:

$$2!\left\langle \mathbf{x}_1',\mathbf{x}_2'\left|\mathbf{x}_1,\mathbf{x}_2\right.\right\rangle = \left\langle 0\left|\hat{\psi}\left(\mathbf{x}_2'\right)\hat{\psi}\left(\mathbf{x}_1'\right)\hat{\psi}^\dagger\left(\mathbf{x}_1\right)\hat{\psi}^\dagger\left(\mathbf{x}_2\right)\right|0\right\rangle \tag{17}$$

$$= \left\langle 0\left|\hat{\psi}\left(\mathbf{x}_2'\right)\left(\delta^3\left(\mathbf{x}_1-\mathbf{x}_1'\right)+\hat{\psi}^\dagger\left(\mathbf{x}_1\right)\hat{\psi}\left(\mathbf{x}_1'\right)\right)\hat{\psi}^\dagger\left(\mathbf{x}_2\right)\right|0\right\rangle \tag{18}$$

$$= \delta^3\left(\mathbf{x}_1-\mathbf{x}_1'\right)\left\langle 0\left|\hat{\psi}\left(\mathbf{x}_2'\right)\hat{\psi}^\dagger\left(\mathbf{x}_2\right)\right|0\right\rangle + \tag{19}$$

$$\left\langle 0\left|\hat{\psi}\left(\mathbf{x}_2'\right)\hat{\psi}^\dagger\left(\mathbf{x}_1\right)\left(\delta^3\left(\mathbf{x}_2-\mathbf{x}_1'\right)+\hat{\psi}^\dagger\left(\mathbf{x}_2\right)\hat{\psi}\left(\mathbf{x}_1'\right)\right)\right|0\right\rangle \tag{20}$$

$$= \delta^3\left(\mathbf{x}_1-\mathbf{x}_1'\right)\left\langle 0\left|\hat{\psi}\left(\mathbf{x}_2'\right)\hat{\psi}^\dagger\left(\mathbf{x}_2\right)\right|0\right\rangle + \tag{21}$$

$$\delta^3\left(\mathbf{x}_2-\mathbf{x}_1'\right)\left\langle 0\left|\hat{\psi}\left(\mathbf{x}_2'\right)\hat{\psi}^\dagger\left(\mathbf{x}_1\right)\right|0\right\rangle + 0 \tag{22}$$

$$= \left(\delta^3\left(\mathbf{x}_1-\mathbf{x}_1'\right)\delta^3\left(\mathbf{x}_2-\mathbf{x}_2'\right)+\delta^3\left(\mathbf{x}_2-\mathbf{x}_1'\right)\delta^3\left(\mathbf{x}_1-\mathbf{x}_2'\right)\right)\left\langle 0\left|0\right.\right\rangle \tag{23}$$

$$= \delta^3\left(\mathbf{x}_1-\mathbf{x}_1'\right)\delta^3\left(\mathbf{x}_2-\mathbf{x}_2'\right)+\delta^3\left(\mathbf{x}_2-\mathbf{x}_1'\right)\delta^3\left(\mathbf{x}_1-\mathbf{x}_2'\right) \tag{24}$$

Again, we've used $\hat{\psi}(\mathbf{x})|0\rangle = 0$ to eliminate terms, and we've assumed the vacuum state is normalized so that $\langle 0|0\rangle = 1$. Note that the result allows for all possible pairings of positions in the bra term (the primed coordinates) with positions in the ket term (unprimed). This is because the particles are assumed to be indistinguishable, so swapping positions has no effect on the outcome. This behaviour is characteristic of bosons (and not fermions, where we need to make different assumptions about the commutators).

In general, the result is as given by Greiner's equation 3.40:

$$\left\langle \mathbf{x}_1',\ldots,\mathbf{x}_n'\left|\mathbf{x}_1,\ldots,\mathbf{x}_n\right.\right\rangle = \frac{1}{n!}\sum_{\text{Permut}} P\left[\delta^3\left(\mathbf{x}_1-\mathbf{x}_1'\right)\ldots\delta^3\left(\mathbf{x}_n-\mathbf{x}_n'\right)\right] \tag{25}$$

The permutation P means that we take all possible pairings of the first term in the argument of the delta functions with the second term. For n positions on each side, this means that the sum contains $n!$ terms.

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