NONRELATIVISTIC FIELD THEORY FOR FERMIONS

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References: W. Greiner & J. Reinhardt, *Field Quantization*, Springer-Verlag (1996), Chapter 3, Section 3.3.

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The nonrelativistic field theory for fermions is quite similar to that for bosons, with the main difference being that instead of commutation relations for the fields, we have anticommutation relations. That is, for the fermion fields $\hat{\psi}(\mathbf{x},t)$ and $\hat{\psi}^{\dagger}(\mathbf{x},t)$ we postulate (as far as I can tell, this is a postulate and not derived from anything else):

In Greiner's eqn 3.55a, a prime is missing from the second **x** on the LHS.

$$\left\{\hat{\psi}(\mathbf{x},t),\hat{\psi}^{\dagger}\left(\mathbf{x}',t\right)\right\} \equiv \hat{\psi}(\mathbf{x},t)\hat{\psi}^{\dagger}\left(\mathbf{x}',t\right)+\hat{\psi}^{\dagger}\left(\mathbf{x}',t\right)\hat{\psi}(\mathbf{x},t) \tag{1}$$

$$=\delta^{3}\left(\mathbf{x}-\mathbf{x}'\right) \tag{2}$$

$$\left\{\hat{\psi}(\mathbf{x},t),\hat{\psi}\left(\mathbf{x}',t\right)\right\}=0 \tag{3}$$

$$\left\{\hat{\psi}^{\dagger}(\mathbf{x},t),\hat{\psi}^{\dagger}\left(\mathbf{x}',t\right)\right\}=0 \tag{4}$$

Using these anticommutators, we can run through the same derivations as we did for bosons to get the corresponding fermion results. As before, we can write the fields as Fourier series:

$$\hat{\psi}(\mathbf{x},t)=\sum_{i}\hat{a}_{i}(t)\,u_{i}(\mathbf{x}) \tag{5}$$

$$\hat{\psi}^{\dagger}(\mathbf{x},t)=\sum_{i}\hat{a}_{i}^{\dagger}(t)\,u_{i}^{*}(\mathbf{x}) \tag{6}$$

where the u_i are assumed to be a complete (in the sense that any other function can be expanded in terms of them) and orthogonal set, so that

$$\int d^{3}x\,u_{i}^{*}(\mathbf{x})\,u_{j}(\mathbf{x})=\delta_{ij} \tag{7}$$

$$\sum_{i}u_{i}(\mathbf{x})\,u_{i}^{*}\left(\mathbf{x}'\right)=\delta^{3}\left(\mathbf{x}-\mathbf{x}'\right) \tag{8}$$

By inserting 5 and 6 into the anticommutators we get

$$\left\{\hat{\psi}(\mathbf{x},t),\hat{\psi}^{\dagger}(\mathbf{x}',t)\right\} = \left\{\sum_i \hat{a}_i(t)u_i(\mathbf{x}),\sum_j \hat{a}_j^{\dagger}(t)u_j^*(\mathbf{x}')\right\} \tag{9}$$

$$= \sum_i \hat{a}_i(t)u_i(\mathbf{x})\sum_j \hat{a}_j^{\dagger}(t)u_j^*(\mathbf{x}')+ \tag{10}$$

$$\sum_j \hat{a}_j^{\dagger}(t)u_j^*(\mathbf{x}')\sum_i \hat{a}_i(t)u_i(\mathbf{x}) \tag{11}$$

$$= \sum_i\sum_j\left(\hat{a}_i(t)\hat{a}_j^{\dagger}(t)+\hat{a}_j^{\dagger}(t)\hat{a}_i(t)\right)u_i(\mathbf{x})u_j^*(\mathbf{x}') \tag{12}$$

$$= \sum_i\sum_j\left\{\hat{a}_i(t),\hat{a}_j^{\dagger}(t)\right\}u_i(\mathbf{x})u_j^*(\mathbf{x}') \tag{13}$$

We require this to be equal to $\delta^3(\mathbf{x}-\mathbf{x}')$ from 2 and we can see this will be true if

$$\left\{\hat{a}_i(t),\hat{a}_j^{\dagger}(t)\right\} = \delta_{ij} \tag{14}$$

since this gives

$$\sum_i\sum_j\left[\hat{a}_i(t),\hat{a}_j^{\dagger}(t)\right]u_i(\mathbf{x})u_j^*(\mathbf{x}') = \sum_i\sum_j \delta_{ij}u_i(\mathbf{x})u_j^*(\mathbf{x}') \tag{15}$$

$$= \sum_i u_i(\mathbf{x})u_i^*(\mathbf{x}') \tag{16}$$

$$= \delta^3(\mathbf{x}-\mathbf{x}') \tag{17}$$

By doing the same calculation with 3 and 4, we get the anticommutators

$$\{\hat{a}_i,\hat{a}_j\} = \left\{\hat{a}_i^{\dagger},\hat{a}_j^{\dagger}\right\} = 0 \tag{18}$$

If we use the same definition of a position space ket as we did for bosons

$$|\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n;t\rangle \equiv \frac{1}{\sqrt{n!}}\hat{\psi}^{\dagger}(\mathbf{x}_1,t)\hat{\psi}^{\dagger}(\mathbf{x}_2,t)\ldots\hat{\psi}^{\dagger}(\mathbf{x}_n,t)|0\rangle \tag{19}$$

then we can see by applying 4 that interchanging any two of the $\hat{\psi}^{\dagger}$ operators will give the negative of the original function. For example, suppose we interchange $\hat{\psi}^{\dagger}(\mathbf{x}_1,t)$ with $\hat{\psi}^{\dagger}(\mathbf{x}_4,t)$. We can do this by moving $\hat{\psi}^{\dagger}(\mathbf{x}_1,t)$ two positions to the right (so we have $\hat{\psi}^{\dagger}(\mathbf{x}_2,t)\hat{\psi}^{\dagger}(\mathbf{x}_3,t)\hat{\psi}^{\dagger}(\mathbf{x}_1,t)\hat{\psi}^{\dagger}(\mathbf{x}_4,t)$). Each move introduces a factor of -1, so two moves will leave the ket unchanged. However, to complete the swap, we need to move $\hat{\psi}^{\dagger}(\mathbf{x}_4,t)$ three positions to the left (so that we end up with $\hat{\psi}^{\dagger}(\mathbf{x}_4,t)\,\hat{\psi}^{\dagger}(\mathbf{x}_2,t)\,\hat{\psi}^{\dagger}(\mathbf{x}_3,t)\,\hat{\psi}^{\dagger}(\mathbf{x}_1,t)$), and this introduces a factor of $(-1)^3 = -1$ so the net result is to multiply the overall ket by -1. In general, if we permute the order of the operators, then the ket is unchanged if it's an even permutation (an even number of swaps) and is multiplied by -1 if it's an odd permutation.

Greiner shows in eqn 3.59 that the number operator, defined as the number of particles in a volume V:

$$\hat{N}_V \equiv \int_V d^3x\,\hat{\psi}^{\dagger}(\mathbf{x},t)\,\hat{\psi}(\mathbf{x},t) \tag{20}$$

satisfies the same commutation relation with the fermion field operator:

$$\left[\hat{N}_V,\hat{\psi}^{\dagger}(\mathbf{x})\right] = \hat{\psi}^{\dagger}(\mathbf{x}) \tag{21}$$

If we consider the time-independent potential case, we can follow through the same derivation for the Hamiltonian $\hat{H}$ as for bosons to get the same result:

$$\hat{H} = \sum_i \hat{a}_i^{\dagger}(t)\,\hat{a}_i(t)\,\epsilon_i \tag{22}$$

where ϵ_i is the energy of the ith state. This result follows because it depends only on the orthonormality of the basis functions u_i and not on the commutation or anticommutation properties.

The time dependence of the $\hat{a}_i(t)$ operators can be worked out the same way as for bosons. Note that time dependence is still determined by the commutator (not the anticommutator) of the operator with the Hamiltonian, so we have

$$i\hbar\dot{\hat{a}}_i(t) = \left[\hat{a}_i(t),\hat{H}\right] \tag{23}$$

$$= \sum_j \left[a_i\hat{(t)},\hat{a}_j^{\dagger}(t)\,\hat{a}_j(t)\right]\epsilon_j \tag{24}$$

In order to make use of the anticommutators 14 and 18, we can use the identity

This can be verified by just writing out the terms.

$$[A,BC] = \{A,B\}\,C - B\,\{A,C\} \tag{25}$$

We then get

$$i\hbar\dot{\hat{a}}_i(t) = \sum_j \left\{\hat{a}_i(t), \hat{a}_j^\dagger(t)\right\}\hat{a}_j(t) - \hat{a}_j^\dagger(t)\left\{\hat{a}_i(t), \hat{a}_j(t)\right\}\epsilon_j \tag{26}$$

$$= \sum_j \delta_{ij}\hat{a}_j(t) - 0 \tag{27}$$

$$= \hat{a}_i(t) \tag{28}$$

The solution is therefore the same as for bosons:

$$\hat{a}_i(t) = \hat{a}_i(0)\, e^{-i\epsilon_i t/\hbar} \tag{29}$$

$$\equiv \hat{a}_i e^{-i\epsilon_i t/\hbar} \tag{30}$$

where Greiner defines the operator $\hat{a}_i$ (without any explicit time dependence) as the value

$$\hat{a}_i \equiv \hat{a}_i(0) \tag{31}$$