NEUTRAL KLEIN-GORDON FIELD - LAGRANGIAN AND HAMILTONIAN

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References: W. Greiner & J. Reinhardt, *Field Quantization*, Springer-Verlag (1996), Chapter 4, Section 4.1.

Post date: 4 Mar 2018.

We've met the Klein-Gordon equation several times before. In Greiner's first use of it in the *Field Quantization* book, he starts with the simplest form, which is for a real field $\phi(\mathbf{x},t)$ and for neutral (uncharged) particles. The Lagrange density (in natural units, where $\hbar=c=1$) is

$$\mathcal{L}(x)=\frac{1}{2}\partial^{\mu}\phi\partial_{\mu}\phi-\frac{1}{2}m^{2}\phi^{2} \tag{1}$$

Here x is the four-vector representing space-time, and we're using the metric

$$ds^{2}=dt^{2}-dx^{2}-dy^{2}-dz^{2} \tag{2}$$

where Greek indices are summed from 0 to 3 and Latin indices from 1 to 3.

Using the Euler-Lagrange equations, we can derive the Klein-Gordon equation from 1. Since ϕ is real, there is only one parameter in the Euler-Lagrange equations, so we have

$$\frac{\partial\mathcal{L}}{\partial\phi}=\frac{\partial}{\partial x_{\mu}}\frac{\partial\mathcal{L}}{\partial\left(\partial^{\mu}\phi\right)} \tag{3}$$

We have

$$\frac{\partial\mathcal{L}}{\partial\phi}=-m^{2}\phi \tag{4}$$

$$\frac{\partial\mathcal{L}}{\partial\left(\partial^{\mu}\phi\right)}=\partial_{\mu}\phi \tag{5}$$

$$\frac{\partial}{\partial x_{\mu}}\frac{\partial\mathcal{L}}{\partial\left(\partial^{\mu}\phi\right)}=\partial^{\mu}\left(\partial_{\mu}\phi\right) \tag{6}$$

$$\equiv\Box\phi \tag{7}$$

where

$$\Box\equiv\partial^{\mu}\partial_{\mu}=\frac{\partial^{2}}{\partial t^{2}}-\nabla^{2} \tag{8}$$

Combining the results we get the K-G equation

$$\left(\Box+m^2\right)\phi=0 \tag{9}$$

Using the notation where a dot indicates a time derivative, we can write 1 as

$$\mathcal{L}(x)=\frac{1}{2}\left(\dot{\phi}^2-\nabla^2\phi\right)-\frac{1}{2}m^2\phi^2 \tag{10}$$

The conjugate momentum is defined as

$$\pi(x)\equiv\frac{\partial\mathcal{L}}{\partial\dot{\phi}(x)}=\dot{\phi}(x) \tag{11}$$

which, using 10, gives a Hamiltonian density of

$$\mathcal{H}=\pi(x)\dot{\phi}(x)-\mathcal{L}(x) \tag{12}$$

$$=\pi^2-\frac{1}{2}\left(\pi^2-\nabla^2\phi\right)+\frac{1}{2}m^2\phi^2 \tag{13}$$

$$=\frac{1}{2}\left(\pi^2+\nabla^2\phi+m^2\phi^2\right) \tag{14}$$

So far, we've taken ϕ to be a wave function, not an operator or a field. In the usual prescription for going from a wave equation to a field theory, we define commutators (or sometimes anticommutators) for the fields and their conjugate momenta. In this case, we define commutation relations as follows.

$$\left[\hat{\phi}(\mathbf{x},t),\hat{\pi}\left(\mathbf{x}',t\right)\right]=i\delta^3\left(\mathbf{x}-\mathbf{x}'\right) \tag{15}$$

$$\left[\hat{\phi}(\mathbf{x},t),\hat{\phi}\left(\mathbf{x}',t\right)\right]=\left[\hat{\pi}(\mathbf{x},t),\hat{\pi}\left(\mathbf{x}',t\right)\right]=0 \tag{16}$$

These commutators aren't Lorentz invariant, since we're assuming that the time t is the same everywhere, so no account is taken of the relativity of time. Greiner promises that we'll come back to this point.

The main job at this point is to use the commutators to derive the equation of motion satisfied by the fields. As Greiner points out, there is in general no guarantee that the fields will satisfy the same equation (in this case, the K-G equation) as the original wave functions.

The usual prescription for finding the time derivative of an operator is to take its commutator with the Hamiltonian. We therefore need the full Hamiltonian, which is the integral of 14 over space:

$$\hat{H}=\frac{1}{2}\int d^3x'\left(\hat{\pi}^2\left(\mathbf{x}',t\right)+\nabla^2\hat{\phi}\left(\mathbf{x}',t\right)+m^2\hat{\phi}^2\left(\mathbf{x}',t\right)\right) \tag{17}$$

We therefore have

$$\dot{\hat{\phi}}(\mathbf{x},t) = -i\left[\hat{\phi}(\mathbf{x},t),\hat{H}\right] \tag{18}$$

Note that $\hat{\phi}(\mathbf{x},t)$ depends on position $\mathbf{x}$ while $\hat{H}$ does not depend on position (as it's integrated over all space). As such, the $\mathbf{x}$ in $\hat{\phi}(\mathbf{x},t)$ is a different variable from the integration variable $\mathbf{x}'$ in the Hamiltonian 17. We can therefore take $\hat{\phi}(\mathbf{x},t)$ inside the integral when calculating the commutator, so we have

$$-i\left[\hat{\phi}(\mathbf{x},t),\hat{H}\right] = -\frac{i}{2}\int d^3x'\left[\hat{\phi}(\mathbf{x},t),\hat{\pi}^2(\mathbf{x}',t)+\nabla^2\hat{\phi}(\mathbf{x}',t)+m^2\hat{\phi}^2(\mathbf{x}',t)\right] \tag{19}$$

We can also take the $\hat{\phi}(\mathbf{x},t)$ inside the derivative ∇, since ∇ operates only on $\mathbf{x}'$, not on $\mathbf{x}$. From 16, the last two terms are therefore zero, and we have

$$\dot{\hat{\phi}}(\mathbf{x},t) = -i\left[\hat{\phi}(\mathbf{x},t),\hat{H}\right] = -\frac{i}{2}\int d^3x'\left[\hat{\phi}(\mathbf{x},t),\hat{\pi}^2(\mathbf{x}',t)\right] \tag{20}$$

$$= -\frac{i}{2}\int d^3x'\left(\hat{\phi}(\mathbf{x},t)\hat{\pi}^2(\mathbf{x}',t)-\hat{\pi}^2(\mathbf{x}',t)\hat{\phi}(\mathbf{x},t)\right) \tag{21}$$

$$= -\frac{i}{2}\int d^3x'\left(i\delta^3(\mathbf{x}-\mathbf{x}')\hat{\pi}(\mathbf{x}',t)+\hat{\pi}(\mathbf{x}',t)\hat{\phi}(\mathbf{x},t)\hat{\pi}(\mathbf{x}',t)+\right. \tag{22}$$

$$\left.i\hat{\pi}(\mathbf{x}',t)\delta^3(\mathbf{x}-\mathbf{x}')-\hat{\pi}(\mathbf{x}',t)\hat{\phi}(\mathbf{x},t)\hat{\pi}(\mathbf{x}',t)\right) \tag{23}$$

$$= -\frac{i}{2}\int d^3x'\left(2i\delta^3(\mathbf{x}-\mathbf{x}')\hat{\pi}(\mathbf{x}',t)\right) \tag{24}$$

$$= \hat{\pi}(\mathbf{x},t) \tag{25}$$

We can do a similar calculation for $\dot{\hat{\pi}}$. Using Greiner's result (again, note that ∇ operates only on $\mathbf{x}'$, not on $\mathbf{x}$):

$$\left[\hat{\pi}(\mathbf{x},t),\nabla\left(\hat{\phi}(\mathbf{x}',t)\right)\right] = -i\nabla\delta^3(\mathbf{x}-\mathbf{x}') \tag{26}$$

we have, doing a similar calculation as in 20

$$\dot{\hat{\pi}}(\mathbf{x},t) = -i\left[\hat{\pi}(\mathbf{x},t),\hat{H}\right] \tag{27}$$

$$= -\frac{i}{2}\int d^3x'\left[\hat{\pi}(\mathbf{x},t),\nabla^2\hat{\phi}\left(\mathbf{x}',t\right)+m^2\hat{\phi}^2\left(\mathbf{x}',t\right)\right] \tag{28}$$

$$= -\frac{i}{2}\int d^3x'\left(-2i\nabla\hat{\phi}\left(\mathbf{x}',t\right)\nabla\delta^3\left(\mathbf{x}-\mathbf{x}'\right)\right)-m^2\hat{\phi}(\mathbf{x},t) \tag{29}$$

The integral can be done by parts, by integrating the $\nabla\delta^3(\mathbf{x}-\mathbf{x}')$ and differentiating the $\nabla\hat{\phi}(\mathbf{x}',t)$. The integrated term is zero because the delta function is zero at infinity (that is, at the limits of the integral). As a result, we get

$$\dot{\hat{\pi}}(\mathbf{x},t) = \int d^3x'\left(\nabla^2\hat{\phi}\left(\mathbf{x}',t\right)\delta^3\left(\mathbf{x}-\mathbf{x}'\right)\right)-m^2\hat{\phi}(\mathbf{x},t) \tag{30}$$

$$= \left(\nabla^2-m^2\right)\hat{\phi}(\mathbf{x},t) \tag{31}$$

Combining 25 and 31 we have

$$\ddot{\hat{\phi}}(\mathbf{x},t) = \dot{\hat{\pi}}(\mathbf{x},t) \tag{32}$$

$$= \left(\nabla^2-m^2\right)\hat{\phi}(\mathbf{x},t) \tag{33}$$

or, in compact notation

$$\left(\Box+m^2\right)\hat{\phi}(\mathbf{x},t) = 0 \tag{34}$$

That is, the field operator $\hat{\phi}(\mathbf{x},t)$ also satisfies the K-G equation.

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