NEUTRAL KLEIN-GORDON FIELD - FOURIER EXPANSION

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: W. Greiner & J. Reinhardt, *Field Quantization*, Springer-Verlag (1996), Chapter 4, Section 4.1.

Post date: 5 Mar 2018.

The real Klein-Gordon field $\hat{\phi}(\mathbf{x},t)$ was shown to satisfy the same Klein-Gordon equation as that satisfied by the wave function, namely

$$\ddot{\hat{\phi}}(\mathbf{x},t)=\left(\nabla^2-m^2\right)\hat{\phi}(\mathbf{x},t) \tag{1}$$

As we did for the nonrelativistic field, we can expand the K-G field using a Fourier integral of the form

$$\hat{\phi}(\mathbf{x},t)=\int d^3p\,u_{\mathbf{p}}(\mathbf{x})\,\hat{a}_{\mathbf{p}}(t) \tag{2}$$

where the $u_{\mathbf{p}}(\mathbf{x})$ are the basis functions and the $\hat{a}_{\mathbf{p}}(t)$ are operators that serve as the Fourier coefficients. One choice of basis is

$$u_{\mathbf{p}}(\mathbf{x})=N_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} \tag{3}$$

so that we have

$$\hat{\phi}(\mathbf{x},t)=\int d^3p\,N_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}}\hat{a}_{\mathbf{p}}(t) \tag{4}$$

where the $N_{\mathbf{p}}$ are normalization constants (for which we assume $N_{\mathbf{p}}=N_{-\mathbf{p}}$).

Using 1 and noting that the time dependence lies entirely within the $\hat{a}_{\mathbf{p}}(t)$ operators, we have

$$\ddot{\hat{\phi}}(\mathbf{x},t)=\int d^3p\,N_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}}\ddot{\hat{a}}_{\mathbf{p}}(t) \tag{5}$$

$$=\int d^3p\,N_{\mathbf{p}}\left(\nabla^2-m^2\right)e^{i\mathbf{p}\cdot\mathbf{x}}\hat{a}_{\mathbf{p}}(t) \tag{6}$$

$$=-\int d^3p\,N_{\mathbf{p}}\left(\mathbf{p}^2+m^2\right)e^{i\mathbf{p}\cdot\mathbf{x}}\hat{a}_{\mathbf{p}}(t) \tag{7}$$

Equating the coefficients of $e^{i\mathbf{p}\cdot\mathbf{x}}$ in the first and last lines, we get

$$\ddot{\hat{a}}_{\mathbf{p}}(t) = -\left(\mathbf{p}^2 + m^2\right)\hat{a}_{\mathbf{p}}(t) \tag{8}$$

for which the general solution is

$$\hat{a}_{\mathbf{p}}(t) = \hat{a}_{\mathbf{p}}^{(1)} e^{-i\omega_{\mathbf{p}}t} + \hat{a}_{\mathbf{p}}^{(2)} e^{i\omega_{\mathbf{p}}t} \tag{9}$$

where

$$\omega_{\mathbf{p}} \equiv \sqrt{\mathbf{p}^2 + m^2} \tag{10}$$

and $\hat{a}_{\mathbf{p}}^{(1)}$ and $\hat{a}_{\mathbf{p}}^{(2)}$ are constant in time, although they do depend on the momentum $\mathbf{p}$.

As we're translating a real classical field ϕ to a quantum field $\hat{\phi}$, the quantum field should be Hermitian, so that $\hat{\phi}^\dagger = \hat{\phi}$. From 4 we have

We're assuming that $N_{\mathbf{p}}$ is real.

$$\begin{aligned} \hat{\phi}^\dagger(\mathbf{x},t) &= \int d^3p\, N_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} \hat{a}_{\mathbf{p}}^\dagger(t) &(11)\\ &= \int d^3p\, N_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} \left(\hat{a}_{\mathbf{p}}^{(1)\dagger} e^{i\omega_{\mathbf{p}}t} + \hat{a}_{\mathbf{p}}^{(2)\dagger} e^{-i\omega_{\mathbf{p}}t}\right) &(12) \end{aligned}$$

If we now propose that

$$\hat{a}_{\mathbf{p}}^{(1)\dagger} = \hat{a}_{-\mathbf{p}}^{(2)} \tag{13}$$

then

$$\hat{a}_{\mathbf{p}}^{(2)\dagger} = \hat{a}_{-\mathbf{p}}^{(1)} \tag{14}$$

so we can write 12 entirely in terms of $\hat{a}_{\mathbf{p}}^{(1)}$, which we'll call just $\hat{a}_{\mathbf{p}}$.

We have

Don't confuse $\hat{a}_{\mathbf{p}}$ (which is constant in time) with the original $\hat{a}_{\mathbf{p}}(t)$ in 9, which is the full time-dependent operator.

$$\begin{aligned} \hat{\phi}^\dagger(\mathbf{x},t) &= \int d^3p\, N_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} \left(\hat{a}_{\mathbf{p}}^\dagger e^{i\omega_{\mathbf{p}}t} + \hat{a}_{-\mathbf{p}} e^{-i\omega_{\mathbf{p}}t}\right) &(15)\\ &= \int d^3p\, N_{\mathbf{p}} \left(\hat{a}_{\mathbf{p}}^\dagger e^{-i(\mathbf{p}\cdot\mathbf{x}-\omega_{\mathbf{p}}t)} + \hat{a}_{-\mathbf{p}} e^{i(-\mathbf{p}\cdot\mathbf{x}-\omega_{\mathbf{p}}t)}\right) &(16) \end{aligned}$$

We can now make the substitution $\mathbf{p} \to -\mathbf{p}$ in the second term. As the integral is over all momentum space, the limits of the integral don't change (well, actually we introduce a minus sign for each component of momentum, but the limits of the integral are inverted which introduces another minus sign, so the two minus signs cancel each other). We therefore get

Here we invoke the assumption that $N_{\mathbf{p}} = N_{-\mathbf{p}}$, which we'll eventually see is true.

$$\hat{\phi}^\dagger(\mathbf{x},t) = \int d^3p\, N_{\mathbf{p}} \left(\hat{a}_{\mathbf{p}}^\dagger e^{-i(\mathbf{p}\cdot\mathbf{x}-\omega_{\mathbf{p}}t)} + \hat{a}_{\mathbf{p}} e^{i(\mathbf{p}\cdot\mathbf{x}-\omega_{\mathbf{p}}t)}\right) \tag{17}$$

From this we can see that

$$\hat{\phi}(\mathbf{x},t) = \left(\hat{\phi}^{\dagger}(\mathbf{x},t)\right)^{\dagger} \tag{18}$$
$$= \int d^3p\, N_{\mathbf{p}} \left(\hat{a}_{\mathbf{p}} e^{+i\left(\mathbf{p}\cdot\mathbf{x}-\omega_{\mathbf{p}}t\right)} + \hat{a}^{\dagger}_{\mathbf{p}} e^{-i\left(\mathbf{p}\cdot\mathbf{x}-\omega_{\mathbf{p}}t\right)}\right) \tag{19}$$
$$= \hat{\phi}^{\dagger}(\mathbf{x},t) \tag{20}$$

so the Hermiticity condition is satisfied.

The conjugate momentum is

$$\hat{\pi}(\mathbf{x},t) = \dot{\hat{\phi}}(\mathbf{x},t) \tag{21}$$
$$= \int d^3p\, N_{\mathbf{p}} \left(-i\omega_{\mathbf{p}}\right) \left(\hat{a}_{\mathbf{p}} e^{+i\left(\mathbf{p}\cdot\mathbf{x}-\omega_{\mathbf{p}}t\right)} - \hat{a}^{\dagger}_{\mathbf{p}} e^{-i\left(\mathbf{p}\cdot\mathbf{x}-\omega_{\mathbf{p}}t\right)}\right) \tag{22}$$

The important feature of this result is that the fields contain contributions with both positive and negative frequencies $\pm\omega_{\mathbf{p}}$. In the nonrelativistic case, the operator $\hat{a}$ had the form

$$\hat{a}_i(t) = \hat{a}_i e^{-i\epsilon_i t/\hbar} \tag{23}$$

where there was only one (positive) energy in the solution. In the K-G case, the two energies arise because the differential equation 8 contains a second derivative with respect to time, which arose from the relativistic condition $E^2 = \mathbf{p}^2 + m^2$, and the fact that the operator equivalent to the energy E is the time derivative $i\frac{\partial}{\partial t}$.