KLEIN-GORDON EQUATION - DERIVATION AND CONTINUITY EQUATIONS

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Reference: W. Greiner: *Relativistic Quantum Mechanics (Wave Equations)*; 3rd Edition, Springer-Verlag (2000); Section 1.2.

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We've already looked at the Klein-Gordon equation, so here I'll just summarize the results in Greiner's notation.

The equation is derived by taking the relativistic equation for the energy of a particle of rest mass m_0

$$E^2 = p^2c^2 + m_0^2c^4 \tag{1}$$

and substituting the quantum operators according to the prescription

$$\hat{E} \to i\hbar\frac{\partial}{\partial t} = \hat{p}^0 \tag{2}$$

$$\hat{p}^i \to i\hbar\frac{\partial}{\partial x_i} \tag{3}$$

Combining the energy and momentum into a four-vector operator and applying the square of this operator to a wave function ψ gives

$$\hat{p}^\mu \hat{p}_\mu \psi = m_0^2c^2\psi \tag{4}$$

This is the Klein-Gordon equation. Written in terms of ordinary derivatives, the equation is

$$\left(\square + \frac{m_0^2c^2}{\hbar^2}\right)\psi = 0 \tag{5}$$

$$\left(\frac{\partial^2}{c^2\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + \frac{m_0^2c^2}{\hbar^2}\right)\psi = 0 \tag{6}$$

Note that Greiner retains the constants c and $\hbar$, rather than setting them equal to 1 as many authors do. Note also that this equation makes no mention of a potential, so strictly speaking it is an equation for a free relativistic particle. Solutions are plane waves of the form

$$\psi = \exp\left(-\frac{i}{\hbar}p_\mu x^\mu\right) \tag{7}$$

$$= \exp\left(-\frac{i}{\hbar}\left(p_0 x^0 - \mathbf{p}\cdot\mathbf{x}\right)\right) \tag{8}$$

$$= \exp\left(\frac{i}{\hbar}\left(\mathbf{p}\cdot\mathbf{x} - Et\right)\right) \tag{9}$$

where we've used the explicit form of the four-momentum and spacetime

$$x^\mu = g^{\mu\nu}x_\nu = \{ct, x, y, z\} \tag{10}$$

$$p_\nu = \left\{\frac{E}{c}, -p_x, -p_y - p_z\right\} \tag{11}$$

Substituting this solution back into 6 leads back to the energy equation 1, which gives E^2 and not just E itself. As a result, the Klein-Gordon equation predicts that a given plane wave solution can have either a positive or negative energy:

$$E = \pm\sqrt{p^2c^2 + m_0^2c^4} \tag{12}$$

In an earlier post, we derived the continuity equation for the Klein-Gordon equation which is

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{j} = 0 \tag{13}$$

In Greiner's notation

$$\rho = \frac{i\hbar}{2m_0c^2}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right) \tag{14}$$

$$\mathbf{j} = -\frac{i\hbar}{2m_0}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) \tag{15}$$

It seems natural to consider $\mathbf{j}$ as a probability current and ρ to be a probability density analogous to the quantity $|\psi|^2$ where ψ here is the wave function that solves the Schrödinger equation in non-relativistic theory. However, although $\mathbf{j}$ in 15 has exactly the same form as the probability current in the Schrödinger equation, the density ρ does not, and in fact is not always positive, since ψ and its derivative could in practice take on values that make ρ negative. Thus we can't interpret the continuity equation 13 as the conservation of probability. This problem, along with the existence of negative energies, were taken to be major problems with the Klein-Gordon equation which led to it being disregarded initially as a valid relativistic equation.

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