

KLEIN-GORDON EQUATION - NONRELATIVISTIC LIMIT

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Reference: W. Greiner: *Relativistic Quantum Mechanics (Wave Equations)*; 3rd Edition, Springer-Verlag (2000); Section 1.3.

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The Klein-Gordon equation is a relativistic wave equation which looks like this

$$\left(\frac{\partial^2}{c^2 \partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + \frac{m_0^2 c^2}{\hbar^2} \right) \psi = 0 \quad (1)$$

Plane wave solutions of this equation are

$$\psi = \exp\left(\frac{i}{\hbar}(\mathbf{p} \cdot \mathbf{x} - Et)\right) \quad (2)$$

We would like to show that in the nonrelativistic limit, this equation reduces to the Schrödinger equation. To do this, we write the wave function ψ as the product of two factors:

$$\psi(\mathbf{r}, t) = \phi(\mathbf{r}, t) e^{-im_0 c^2 t / \hbar} \quad (3)$$

That is, we have split the portion of the energy due to the rest energy of the particle off into its own factor, so that the energy contained in the ϕ factor is the kinetic energy $E' = E - m_0 c^2$. For a nonrelativistic particle, we would expect $E' \ll m_0 c^2$. If we operate on ϕ with the energy operator $\hat{E} = i\hbar \frac{\partial}{\partial t}$, we would then expect that this would give

$$\left| i\hbar \frac{\partial \phi}{\partial t} \right| \approx E' \phi \ll m_0 c^2 \phi \quad (4)$$

Taking the time derivative of 3 we therefore have, using 4

$$\frac{\partial \psi}{\partial t} = \left(\frac{\partial \phi}{\partial t} - i \frac{m_0 c^2}{\hbar} \phi \right) e^{-im_0 c^2 t / \hbar} \quad (5)$$

$$\approx -i \frac{m_0 c^2}{\hbar} \phi e^{-im_0 c^2 t / \hbar} \quad (6)$$

We now need to take the second derivative of ψ , and at this point, Greiner appears to make the assumption that we can neglect $\frac{\partial^2 \phi}{\partial t^2}$. It's not entirely

clear to me how we can justify this assumption, but if we go with it, we get, using the product rule on 5

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial}{\partial t} \left[\left(\frac{\partial \phi}{\partial t} - i \frac{m_0 c^2}{\hbar} \phi \right) e^{-im_0 c^2 t / \hbar} \right] \quad (7)$$

$$\approx -i \frac{m_0 c^2}{\hbar} \frac{\partial \phi}{\partial t} e^{-im_0 c^2 t / \hbar} - i \frac{m_0 c^2}{\hbar} \left(\frac{\partial \phi}{\partial t} - i \frac{m_0 c^2}{\hbar} \phi \right) e^{-im_0 c^2 t / \hbar} \quad (8)$$

$$= - \left(i \frac{2m_0 c^2}{\hbar} \frac{\partial \phi}{\partial t} + \frac{m_0^2 c^4}{\hbar^2} \phi \right) e^{-im_0 c^2 t / \hbar} \quad (9)$$

Inserting this back into 1 and cancelling terms gives

$$-i \frac{2m_0}{\hbar} \frac{\partial \phi}{\partial t} - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \phi = 0 \quad (10)$$

Rearranging terms and multiplying through by $-\frac{\hbar^2}{2m_0}$ we get

$$i\hbar \frac{\partial \phi}{\partial t} = -\frac{\hbar^2}{2m_0} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \phi \quad (11)$$

$$= -\frac{\hbar^2}{2m_0} \nabla^2 \phi \quad (12)$$

This is just the Schrödinger equation for a free particle (and zero spin), where ϕ has taken on the role of the wave function.