KLEIN-GORDON EQUATION - NONRELATIVISTIC LIMIT

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Reference: W. Greiner: *Relativistic Quantum Mechanics (Wave Equations)*; 3rd Edition, Springer-Verlag (2000); Section 1.3.
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The Klein-Gordon equation is a relativistic wave equation which looks like this

$$\left(\frac{\partial^2}{c^2\partial t^2}-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial z^2}+\frac{m_0^2c^2}{\hbar^2}\right)\psi=0 \tag{1}$$

Plane wave solutions of this equation are

$$\psi=\exp\left(\frac{i}{\hbar}\left(\mathbf{p}\cdot\mathbf{x}-Et\right)\right) \tag{2}$$

We would like to show that in the nonrelativistic limit, this equation reduces to the Schrödinger equation. To do this, we write the wave function ψ as the product of two factors:

$$\psi\left(\mathbf{r},t\right)=\phi\left(\mathbf{r},t\right)e^{-im_0c^2t/\hbar} \tag{3}$$

That is, we have split the portion of the energy due to the rest energy of the particle off into its own factor, so that the energy contained in the ϕ factor is the kinetic energy $E'=E-m_0c^2$. For a nonrelativistic particle, we would expect $E'\ll m_0c^2$. If we operate on ϕ with the energy operator $\hat{E}=i\hbar\frac{\partial}{\partial t}$, we would then expect that this would give

$$\left|i\hbar\frac{\partial\phi}{\partial t}\right|\approx E'\phi\ll m_0c^2\phi \tag{4}$$

Taking the time derivative of 3 we therefore have, using 4

$$\frac{\partial\psi}{\partial t}=\left(\frac{\partial\phi}{\partial t}-i\frac{m_0c^2}{\hbar}\phi\right)e^{-im_0c^2t/\hbar} \tag{5}$$

$$\approx-i\frac{m_0c^2}{\hbar}\phi e^{-im_0c^2t/\hbar} \tag{6}$$

We now need to take the second derivative of ψ, and at this point, Greiner appears to make the assumption that we can neglect $\frac{\partial^2\phi}{\partial t^2}$. It's not entirely clear to me how we can justify this assumption, but if we go with it, we get, using the product rule on 5

$$\frac{\partial^2\psi}{\partial t^2} = \frac{\partial}{\partial t}\left[\left(\frac{\partial\phi}{\partial t} - i\frac{m_0c^2}{\hbar}\phi\right)e^{-im_0c^2t/\hbar}\right] \tag{7}$$

$$\approx -i\frac{m_0c^2}{\hbar}\frac{\partial\phi}{\partial t}e^{-im_0c^2t/\hbar} - i\frac{m_0c^2}{\hbar}\left(\frac{\partial\phi}{\partial t} - i\frac{m_0c^2}{\hbar}\phi\right)e^{-im_0c^2t/\hbar} \tag{8}$$

$$= -\left(i\frac{2m_0c^2}{\hbar}\frac{\partial\phi}{\partial t} + \frac{m_0^2c^4}{\hbar^2}\phi\right)e^{-im_0c^2t/\hbar} \tag{9}$$

Inserting this back into 1 and cancelling terms gives

$$-i\frac{2m_0}{\hbar}\frac{\partial\phi}{\partial t} - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = 0 \tag{10}$$

Rearranging terms and multiplying through by $-\frac{\hbar^2}{2m_0}$ we get

$$i\hbar\frac{\partial\phi}{\partial t} = -\frac{\hbar^2}{2m_0}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi \tag{11}$$

$$= -\frac{\hbar^2}{2m_0}\nabla^2\phi \tag{12}$$

This is just the Schrödinger equation for a free particle (and zero spin), where ϕ has taken on the role of the wave function.