KLEIN-GORDON EQUATION - CHARGED PARTICLES

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Reference: W. Greiner: *Relativistic Quantum Mechanics (Wave Equations)*; 3rd Edition, Springer-Verlag (2000); Section 1.4.

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The Klein-Gordon equation is a relativistic wave equation which looks like this

$$\left(\frac{\partial^2}{c^2\partial t^2}-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial z^2}+\frac{m_0^2c^2}{\hbar^2}\right)\psi=0 \tag{1}$$

or in four-vector form

$$\left(p^\mu p_\mu+\frac{m_0^2c^2}{\hbar^2}\right)\psi=0 \tag{2}$$

Solutions of this equation satisfy the continuity relation

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\mathbf{j}=0 \tag{3}$$

where

$$\rho=\frac{i\hbar}{2m_0c^2}\left(\psi^*\frac{\partial\psi}{\partial t}-\psi\frac{\partial\psi^*}{\partial t}\right) \tag{4}$$

$$\mathbf{j}=-\frac{i\hbar}{2m_0}\left(\psi^*\nabla\psi-\psi\nabla\psi^*\right) \tag{5}$$

We can't interpret ρ as a probability density because it's not positive definite, and a negative probability doesn't make sense. However, if we multiply ρ and $\mathbf{j}$ by the elementary charge e, we can define charge and current densities:

$$\rho'=\frac{i\hbar e}{2m_0c^2}\left(\psi^*\frac{\partial\psi}{\partial t}-\psi\frac{\partial\psi^*}{\partial t}\right) \tag{6}$$

$$\mathbf{j}'=-\frac{i\hbar e}{2m_0}\left(\psi^*\nabla\psi-\psi\nabla\psi^*\right) \tag{7}$$

Because free wave solutions allow both a positive and negative energy for a given momentum $\mathbf{p}$, there are two solutions for each momentum:

$$\psi_{(\pm)} = A_{(\pm)} e^{i\left(\mathbf{p}\cdot\mathbf{x}\mp|E_p|t\right)/\hbar} \tag{8}$$

where the two constants $A_{(\pm)}$ are normalization constants. Inserting these solutions into the charge density 6 we get

$$\rho' = \frac{i\hbar e}{2m_0c^2}\left(\psi^*_{(\pm)}\left(\mp\frac{i}{\hbar}|E_p|\psi_{(\pm)}\right) - \psi_{(\pm)}\left(\pm\frac{i}{\hbar}|E_p|\psi^*_{(\pm)}\right)\right) \tag{9}$$

$$= \frac{e|E_p|}{2m_0c^2}\left(\psi^*_{(\pm)}\left(\pm\psi_{(\pm)}\right) + \psi_{(\pm)}\left(\pm\psi^*_{(\pm)}\right)\right) \tag{10}$$

$$= \pm\frac{e|E_p|}{m_0c^2}\psi^*_{(\pm)}\psi_{(\pm)} \tag{11}$$

Thus we can interpret ρ' as a charge density if the solutions $\psi_{(+)}$ represents particles with positive charge ($+e$) and $\psi_{(-)}$ represents particles with negative charge.

Greiner then shows that if restrict the system to a cube of side length L (effectively a 3-d infinite square well), then the positive and negative charge solutions are

$$\psi_{n(\pm)} = \sqrt{\frac{m_0c^2}{L^3E_{p_n}}}e^{i\left(\mathbf{p}_n\cdot\mathbf{x}\mp|E_{p_n}|t\right)/\hbar} \tag{12}$$

where **n** represents a set of 3 integers, one for each coordinate direction x, y and z. The normalization constant is the same for both solutions:

$$A_{(\pm)} = A = \sqrt{\frac{m_0c^2}{L^3E_{p_n}}} \tag{13}$$

The general solutions for positive and negative particles are then linear combinations of the $\psi_{n(+)}$ or $\psi_{n(-)}$ wave functions.

With the charge density defined by 6, we can guess that a solution representing neutral (zero charge) particles might be one where $\rho' = 0$. This will happen if ψ is real, since then the quantity in parentheses in 6 is zero. From 12 we see that

$$\psi_{n(+)}(\mathbf{p}_n) = \psi^*_{n(-)}(-\mathbf{p}_n) \tag{14}$$

Therefore, the linear combination

$$\psi_{n(0)} = \frac{1}{\sqrt{2}}\left(\psi_{n(+)}(\mathbf{p}_n) + \psi_{n(-)}(-\mathbf{p}_n)\right) \tag{15}$$

is real, and Greiner shows in eq 1.46 that this works out to

$$\psi_{n(0)} = \sqrt{\frac{2m_0c^2}{L^3 E_{p_n}}} \cos\left(\frac{\mathbf{p}_n \cdot \mathbf{x} - E_{p_n} t}{\hbar}\right) \tag{16}$$

From 7, we see that the charge current is also zero if ψ is real.

Thus the solutions of the Klein-Gordon equation can be used to describe particles with positive, negative or zero charge.

In Greiner's Example 1.1, he shows that for a complex solution ϕ (which can be used to represent charged particles), we can decompose it into its real and imaginary parts

$$\phi(x) = \frac{1}{\sqrt{2}}\left(\phi_1(x) + i\phi_2(x)\right) \tag{17}$$

where ϕ_1 and ϕ_2 are real functions. Since 1 is a linear differential equation, the real and imaginary parts are each solutions in their own right. By interpreting 6 as the charge density, we can find the total charge in the system by integrating:

$$Q = \frac{i\hbar e}{2m_0c^2}\int d^3x\left(\phi^*\dot{\phi} - \phi\dot{\phi}^*\right) \tag{18}$$

If we replace ϕ by ϕ^*, we get an equal but opposite charge.

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