

KLEIN-GORDON EQUATION - CHARGED PARTICLES

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Reference: W. Greiner: *Relativistic Quantum Mechanics (Wave Equations)*; 3rd Edition, Springer-Verlag (2000); Section 1.4.

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The Klein-Gordon equation is a relativistic wave equation which looks like this

$$\left(\frac{\partial^2}{c^2 \partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + \frac{m_0^2 c^2}{\hbar^2} \right) \psi = 0 \quad (1)$$

or in four-vector form

$$\left(p^\mu p_\mu + \frac{m_0^2 c^2}{\hbar^2} \right) \psi = 0 \quad (2)$$

Solutions of this equation satisfy the continuity relation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0 \quad (3)$$

where

$$\rho = \frac{i\hbar}{2m_0 c^2} \left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right) \quad (4)$$

$$\mathbf{j} = -\frac{i\hbar}{2m_0} (\psi^* \nabla \psi - \psi \nabla \psi^*) \quad (5)$$

We can't interpret ρ as a probability density because it's not positive definite, and a negative probability doesn't make sense. However, if we multiply ρ and \mathbf{j} by the elementary charge e , we can define charge and current densities:

$$\rho' = \frac{i\hbar e}{2m_0 c^2} \left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right) \quad (6)$$

$$\mathbf{j}' = -\frac{i\hbar e}{2m_0} (\psi^* \nabla \psi - \psi \nabla \psi^*) \quad (7)$$

Because free wave solutions allow both a positive and negative energy for a given momentum \mathbf{p} , there are two solutions for each momentum:

$$\Psi_{(\pm)} = A_{(\pm)} e^{i(\mathbf{p} \cdot \mathbf{x} \mp |E_p|t)/\hbar} \quad (8)$$

where the two constants $A_{(\pm)}$ are normalization constants. Inserting these solutions into the charge density 6 we get

$$\rho' = \frac{i\hbar e}{2m_0c^2} \left(\Psi_{(\pm)}^* \left(\mp \frac{i}{\hbar} |E_p| \Psi_{(\pm)} \right) - \Psi_{(\pm)} \left(\pm \frac{i}{\hbar} |E_p| \Psi_{(\pm)}^* \right) \right) \quad (9)$$

$$= \frac{e |E_p|}{2m_0c^2} \left(\Psi_{(\pm)}^* (\pm \Psi_{(\pm)}) + \Psi_{(\pm)} (\pm \Psi_{(\pm)}^*) \right) \quad (10)$$

$$= \pm \frac{e |E_p|}{m_0c^2} \Psi_{(\pm)}^* \Psi_{(\pm)} \quad (11)$$

Thus we can interpret ρ' as a charge density if the solutions $\Psi_{(+)}$ represents particles with positive charge ($+e$) and $\Psi_{(-)}$ represents particles with negative charge.

Greiner then shows that if restrict the system to a cube of side length L (effectively a 3-d infinite square well), then the positive and negative charge solutions are

$$\Psi_{n(\pm)} = \sqrt{\frac{m_0c^2}{L^3 E_{p_n}}} e^{i(\mathbf{p}_n \cdot \mathbf{x} \mp |E_{p_n}|t)/\hbar} \quad (12)$$

where \mathbf{n} represents a set of 3 integers, one for each coordinate direction x , y and z . The normalization constant is the same for both solutions:

$$A_{(\pm)} = A = \sqrt{\frac{m_0c^2}{L^3 E_{p_n}}} \quad (13)$$

The general solutions for positive and negative particles are then linear combinations of the $\Psi_{n(+)}$ or $\Psi_{n(-)}$ wave functions.

With the charge density defined by 6, we can guess that a solution representing neutral (zero charge) particles might be one where $\rho' = 0$. This will happen if Ψ is real, since then the quantity in parentheses in 6 is zero. From 12 we see that

$$\Psi_{n(+)}(\mathbf{p}_n) = \Psi_{n(-)}^*(-\mathbf{p}_n) \quad (14)$$

Therefore, the linear combination

$$\Psi_{n(0)} = \frac{1}{\sqrt{2}} (\Psi_{n(+)}(\mathbf{p}_n) + \Psi_{n(-)}(-\mathbf{p}_n)) \quad (15)$$

is real, and Greiner shows in eq 1.46 that this works out to

$$\psi_{n(0)} = \sqrt{\frac{2m_0c^2}{L^3E_{p_n}}} \cos\left(\frac{\mathbf{p}_n \cdot \mathbf{x} - E_{p_n}t}{\hbar}\right) \quad (16)$$

From 7, we see that the charge current is also zero if ψ is real.

Thus the solutions of the Klein-Gordon equation can be used to describe particles with positive, negative or zero charge.

In Greiner's Example 1.1, he shows that for a complex solution ϕ (which can be used to represent charged particles), we can decompose it into its real and imaginary parts

$$\phi(x) = \frac{1}{\sqrt{2}} (\phi_1(x) + i\phi_2(x)) \quad (17)$$

where ϕ_1 and ϕ_2 are real functions. Since 1 is a linear differential equation, the real and imaginary parts are each solutions in their own right. By interpreting 6 as the charge density, we can find the total charge in the system by integrating:

$$Q = \frac{i\hbar e}{2m_0c^2} \int d^3x (\phi^* \dot{\phi} - \phi \dot{\phi}^*) \quad (18)$$

If we replace ϕ by ϕ^* , we get an equal but opposite charge.

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