ENERGY-MOMENTUM TENSOR FOR A GENERAL LAGRANGE DENSITY

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Reference: W. Greiner: *Relativistic Quantum Mechanics (Wave Equations)*; 3rd Edition, Springer-Verlag (2000); Section 1.5; Example 1.3.

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In Example 1.3, Greiner shows how invariance under an infinistesimal spacetime translation leads to the energy-momentum tensor and its continuity equations. Although most of his derivation is easy to follow, there are a few points that need clarification.

We begin with the translations

$$x'_\mu = x_\mu + \varepsilon_\mu \tag{1}$$

The variation in the spacetime coordinates is therefore

$$\delta x_\mu = x'_\mu - x_\mu = \varepsilon_\mu \tag{2}$$

Greiner then states that corresponding variation in the Lagrangian (strictly, the Lagrangian density) is

$$\delta\mathscr{L} = \frac{\partial\mathscr{L}}{\partial x_\mu}\delta x_\mu = \varepsilon_\mu \frac{\partial\mathscr{L}}{\partial x_\mu} \tag{3}$$

At this point, he invokes the condition that the Lagrangian is translationally invariant, so we can write it as

$$\mathscr{L} = \mathscr{L}\left(\psi_\sigma, \psi_{\sigma,\mu}\right) \tag{4}$$

A comma in the subscript indicates a derivative with respect to the following index, so that $\psi_{\sigma,\mu} = \partial\psi_\sigma/\partial x_\mu$.

That is, $\mathscr{L}$ doesn't depend explicitly on x_μ. However, it would seem to me that if the Lagrangian is translationally invariant so that it doesn't depend explicitly on x_μ, then $\frac{\partial\mathscr{L}}{\partial x_\mu} = 0$, which would give $\delta\mathscr{L} = 0$ from 3. If we take $\frac{\partial\mathscr{L}}{\partial x_\mu} = 0$, though, this would imply that the Lagrangian is constant over all time and space, which doesn't seem valid either. I suspect that 3 is saying that $\delta\mathscr{L}$ is the amount by which $\mathscr{L}$ changes if we move from x_μ to $x_\mu + \varepsilon_\mu$. Comment welcome.

Anyway, from 4, we can write the variation as

$$\delta\mathscr{L}=\sum_{\sigma}\left[\frac{\partial\mathscr{L}}{\partial\psi_{\sigma}}\delta\psi_{\sigma}+\frac{\partial\mathscr{L}}{\partial\left(\psi_{\sigma,\mu}\right)}\delta\psi_{\sigma,\mu}\right] \tag{5}$$

By equating 3 and 5, Greiner shows that in his equations (4) through (8) that we get the condition

The summation convention is used here for μ since it is a sum over coordinate indexes. The sum over σ is written explicitly since it is a sum over independent fields.

$$\varepsilon_{\mu}\frac{\partial\mathscr{L}}{\partial x_{\mu}}=\frac{\partial}{\partial x_{\mu}}\left[\sum_{\sigma}\frac{\partial\mathscr{L}}{\partial\left(\psi_{\sigma,\mu}\right)}\varepsilon_{\nu}\psi_{\sigma,\nu}\right] \tag{6}$$

The trick now is to notice that the variations ε_{μ} are arbitrary. This means that in 6 the coefficients of each component ε_{μ} must be equal on both sides of the equation. That is, if we consider ε_{0}, then we must have

$$\frac{\partial\mathscr{L}}{\partial x_{0}}=\frac{\partial}{\partial x_{\mu}}\left[\sum_{\sigma}\frac{\partial\mathscr{L}}{\partial\left(\psi_{\sigma,\mu}\right)}\psi_{\sigma,0}\right] \tag{7}$$

Similarly for ε_{1} we must have

$$\frac{\partial\mathscr{L}}{\partial x_{1}}=\frac{\partial}{\partial x_{\mu}}\left[\sum_{\sigma}\frac{\partial\mathscr{L}}{\partial\left(\psi_{\sigma,\mu}\right)}\psi_{\sigma,1}\right] \tag{8}$$

and so on for the other two indexes. In general, then, we must have

$$\frac{\partial\mathscr{L}}{\partial x_{\nu}}=\frac{\partial}{\partial x_{\mu}}\left[\sum_{\sigma}\frac{\partial\mathscr{L}}{\partial\left(\psi_{\sigma,\mu}\right)}\psi_{\sigma,\nu}\right] \tag{9}$$

To write a general formula, we need to recall the notation used by Greiner. The covariant form of the coordinates is

$$x_{\mu}=\{ct,-x,-y,-z\} \tag{10}$$

The metric tensor is

$$g_{\mu\nu}=\left[\begin{array}{cccc}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{array}\right] \tag{11}$$

If we try to write the LHS of 9 as

$$\frac{\partial\mathscr{L}}{\partial x_{\nu}}=g_{\mu\nu}\frac{\partial\mathscr{L}}{\partial x_{\mu}} \tag{12}$$

we run into a problem, since the sign on the RHS is wrong for $\nu=i=1,2,3$. That is

$$g_{\mu i}\frac{\partial \mathscr{L}}{\partial x_\mu} = -\delta_{\mu i}\frac{\partial \mathscr{L}}{\partial x_\mu} = -\frac{\partial \mathscr{L}}{\partial x_i} \tag{13}$$

However, we can fix this by raising the index ν on the RHS of 9, which changes the sign of the $\psi_{\sigma,\nu}$ factor. Notice that the position of the ν index in the $\frac{\partial \psi_\sigma}{\partial x_\nu}$ term Greiner's (8) has changed from lower to upper in eqn (10), where it becomes $\frac{\partial \psi_\sigma}{\partial x^\nu}$. Doing this means we can write 9 as

$$g_{\mu\nu}\frac{\partial \mathscr{L}}{\partial x_\mu} = \frac{\partial}{\partial x_\mu}\left[\sum_\sigma \frac{\partial \mathscr{L}}{\partial\left(\psi_{\sigma,\mu}\right)}\frac{\partial \psi_\sigma}{\partial x^\nu}\right] \tag{14}$$

Putting everything on the RHS we get

$$\frac{\partial}{\partial x_\mu}\left[-g_{\mu\nu}\mathscr{L} + \sum_\sigma \frac{\partial \mathscr{L}}{\partial\left(\psi_{\sigma,\mu}\right)}\frac{\partial \psi_\sigma}{\partial x^\nu}\right] = 0 \tag{15}$$

The quantity in brackets is the energy-momentum tensor

$$T_{\mu\nu} = -g_{\mu\nu}\mathscr{L} + \sum_\sigma \frac{\partial \mathscr{L}}{\partial\left(\psi_{\sigma,\mu}\right)}\frac{\partial \psi_\sigma}{\partial x^\nu} \tag{16}$$

and it satisfies the continuity equation

$$\frac{\partial}{\partial x_\mu}T_{\mu\nu} = 0 \tag{17}$$

Greiner now examines the $T_{0\nu}$ components in eqns (11) through (17). He does this by using the Hamiltonian density

$$\mathscr{H} = \sum_\sigma \pi_\sigma \dot{\psi}_\sigma - \mathscr{L} \tag{18}$$

and the conjugate momentum

$$\pi_\sigma \equiv \frac{\partial \mathscr{L}}{\partial \dot{\psi}_\sigma} \tag{19}$$

Plugging these into 16 with $\mu = 0$ and integrating over space we get the quantity

$$p_\nu = \int d^3x\, T_{0\nu} \tag{20}$$

$$= \int d^3x \left[\sum_\sigma \frac{\partial \mathscr{L}}{\partial \dot{\psi}_\sigma}\frac{\partial \psi_\sigma}{\partial x^\nu} - g_{0\nu}\mathscr{L}\right] \tag{21}$$

$$= \int d^3x \left[\sum_\sigma \pi_\sigma \frac{\partial \psi_\sigma}{\partial x^\nu} - \delta_{0\nu}\mathscr{L}\right] \tag{22}$$

From17 we have

$$\frac{\partial}{\partial x_0}T_{0\nu} = -\frac{\partial}{\partial x_i}T_{i\nu} \tag{23}$$

Taking the derivative of 20, we get

$$\frac{\partial p_\nu}{\partial x_0} = \int d^3x\, \frac{\partial}{\partial x_0}T_{0\nu} \tag{24}$$

$$= -\int d^3x\, \frac{\partial}{\partial x_i}T_{i\nu} \tag{25}$$

As the sum over i in the last line is over the spatial coordinates, the quantity $\frac{\partial}{\partial x_i}T_{i\nu}$ is a divergence. We can then use Gauss's theorem to convert the volume integral to a surface integral, and if we make the usual assumption that the integrand goes to zero fast enough as the surface goes to infinity, the integral comes out to zero. Thus we have

$$\frac{\partial p_\nu}{\partial x_0} = \frac{\partial p_\nu}{c\,\partial t} = 0 \tag{26}$$

and the quantities p_ν are conserved over time. Furthermore

$$T_{00} = \sum_\sigma \pi_\sigma \dot{\psi}_\sigma - \mathscr{L} = \mathscr{H} \tag{27}$$

so the integral of T_{00} over space gives the total energy H.

The interpretation of the spatial components p_i as momentum components is a bit trickier, since they were derived from the conjugate momenta π_σ. See the earlier post for a discussion of the meaning of conjugate momentum in a field theory.

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