

## SCHRÖDINGER EQUATION: LAGRANGE DENSITY & ENERGY-MOMENTUM TENSOR

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Reference: W. Greiner: *Relativistic Quantum Mechanics (Wave Equations)*; 3rd Edition, Springer-Verlag (2000); Section 1.5; Exercise 1.4.

Post date: 4 Dec 2017.

The Lagrange density for the Schrödinger equation is

$$\mathcal{L}(\psi, \nabla\psi, \dot{\psi}) = i\hbar\psi^*\dot{\psi} - \frac{\hbar^2}{2m}\nabla\psi^* \cdot \nabla\psi - V(\mathbf{x}, t)\psi^*\psi \quad (1)$$

Also in the earlier post, we showed that the Hamiltonian density works out to

$$\mathcal{H} = \frac{\hbar^2}{2m}\nabla\psi^* \cdot \nabla\psi + V(\mathbf{x}, t)\psi^*\psi \quad (2)$$

Integrating this over space gives the result

$$H = \int d^3x \left[ -\frac{\hbar^2}{2m}\psi^*\nabla^2\psi + V(\mathbf{x}, t)\psi^*\psi \right] \quad (3)$$

$$= \int d^3x \psi^* \left[ -\frac{\hbar^2}{2m}\nabla^2\psi + V(\mathbf{x}, t)\psi \right] \quad (4)$$

$$= \langle \psi | \hat{H} | \psi \rangle \quad (5)$$

We can work out the energy-momentum tensor for the Schrödinger Lagrangian.

$$T_{\mu\nu} = -g_{\mu\nu}\mathcal{L} + \sum_{\sigma} \frac{\partial\mathcal{L}}{\partial(\psi_{\sigma,\mu})} \frac{\partial\psi_{\sigma}}{\partial x^{\nu}} \quad (6)$$

In this case, there are two independent fields:  $\psi$  and  $\psi^*$ , so the tensor comes out to (raising the second index):

$$T_{\mu\nu} = -g_{\mu\nu}\mathcal{L} + \frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_{\mu})} \frac{\partial\psi}{\partial x^{\nu}} + \frac{\partial\mathcal{L}}{\partial(\partial\psi^*/\partial x_{\mu})} \frac{\partial\psi^*}{\partial x^{\nu}} \quad (7)$$

The form in Greiner's eqn (10) raises the  $\nu$  index. It also interchanges  $\mu$  and  $\nu$  on the RHS, so he states

$$T_{\mu}^{\nu} = -\delta_{\mu}^{\nu}\mathcal{L} + \frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x^{\nu})}\frac{\partial\psi}{\partial x^{\mu}} + \frac{\partial\mathcal{L}}{\partial(\partial\psi^{*}/\partial x^{\nu})}\frac{\partial\psi^{*}}{\partial x^{\mu}} \quad (8)$$

Although it's not obvious from the definition 6, it seems that in most cases the energy-momentum tensor is indeed symmetric (see, for example, Wikipedia). For the Schrödinger Lagrangian 1, we can show by direct calculation that  $T_{\mu\nu}$  is symmetric for spatial components. From 6

$$T_{ij} = -g_{ij}\mathcal{L} + \frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_i)}\frac{\partial\psi}{\partial x^j} + \frac{\partial\mathcal{L}}{\partial(\partial\psi^{*}/\partial x_i)}\frac{\partial\psi^{*}}{\partial x^j} \quad (9)$$

$$= -g_{ij}\mathcal{L} - \frac{\hbar^2}{2m} \left[ -\frac{\partial\psi^{*}}{\partial x^i}\frac{\partial\psi}{\partial x^j} - \frac{\partial\psi}{\partial x^i}\frac{\partial\psi^{*}}{\partial x^j} \right] \quad (10)$$

$$= T_{ji} \quad (11)$$

However, it is not symmetric for mixed time and space coordinates. For example, using 6

The minus signs inside the brackets come from  $\frac{\partial\psi}{\partial x_i} = -\frac{\partial\psi}{\partial x^i}$  for spatial coordinates.

$$T_{01} = \frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_0)}\frac{\partial\psi}{\partial x^1} + \frac{\partial\mathcal{L}}{\partial(\partial\psi^{*}/\partial x_0)}\frac{\partial\psi^{*}}{\partial x^1} \quad (12)$$

$$= \frac{\partial\mathcal{L}}{\partial\dot{\psi}}\frac{\partial\psi}{\partial x^1} + \frac{\partial\mathcal{L}}{\partial\dot{\psi}^{*}}\frac{\partial\psi^{*}}{\partial x^1} \quad (13)$$

$$= -\frac{\hbar}{2i} \left( \psi^{*}\frac{\partial\psi}{\partial x^1} - \psi\frac{\partial\psi^{*}}{\partial x^1} \right) \quad (14)$$

$$T_{10} = \frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_1)}\frac{\partial\psi}{\partial x^0} + \frac{\partial\mathcal{L}}{\partial(\partial\psi^{*}/\partial x_1)}\frac{\partial\psi^{*}}{\partial x^0} \quad (15)$$

$$= -\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x)}\dot{\psi} + \frac{\partial\mathcal{L}}{\partial(\partial\psi^{*}/\partial x)}\dot{\psi}^{*} \quad (16)$$

$$= -\frac{\hbar^2}{2m} \left( -\frac{\partial\psi^{*}}{\partial x^1}\dot{\psi} - \frac{\partial\psi}{\partial x^1}\dot{\psi}^{*} \right) \quad (17)$$

Greiner then shows (using his version of  $T_{\mu}^{\nu}$  in 8) in his eqn (11) that

$$T_0^0 = \frac{\hbar^2}{2m}\nabla\psi^{*}\cdot\nabla\psi + V(\mathbf{x},t)\psi^{*}\psi = \mathcal{H} \quad (18)$$

Thus  $T_0^0$  is the energy density.

Greiner also calculates the energy flux, which he says is defined by

$$\mathbf{S} = e_1 T_0^1 + e_2 T_0^2 + e_3 T_0^3 \quad (19)$$

$$= -\frac{\hbar^2}{2m_0} (\psi^* \nabla \psi + \psi \nabla \psi^*) \quad (20)$$

Also, the momentum density

$$\mathbf{p} = e_1 T_1^0 + e_2 T_2^0 + e_3 T_3^0 \quad (21)$$

$$= \frac{i\hbar}{2} (\psi^* \nabla \psi - \psi \nabla \psi^*) \quad (22)$$

This is  $-m\mathbf{J}$ , where  $\mathbf{J}$  is the 3-d probability current.

Using the original definition of  $T_{\mu\nu}$  in 6, the definitions of  $\mathbf{S}$  and  $\mathbf{p}$  would be swapped. I'm not sure whether the equations in Greiner are typos or whether there's something deeper going on here.