SCHRÖDINGER EQUATION: LAGRANGE DENSITY & ENERGY-MOMENTUM TENSOR

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Reference: W. Greiner: *Relativistic Quantum Mechanics (Wave Equations)*; 3rd Edition, Springer-Verlag (2000); Section 1.5; Exercise 1.4.

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The Lagrange density for the Schrödinger equation is

$$\mathcal{L}\left(\psi,\nabla\psi,\dot{\psi}\right)=i\hbar\psi^{*}\dot{\psi}-\frac{\hbar^{2}}{2m}\nabla\psi^{*}\cdot\nabla\psi-V\left(\mathbf{x},t\right)\psi^{*}\psi \tag{1}$$

Also in the earlier post, we showed that the Hamiltonian density works out to

$$\mathcal{H}=\frac{\hbar^{2}}{2m}\nabla\psi^{*}\cdot\nabla\psi+V\left(\mathbf{x},t\right)\psi^{*}\psi \tag{2}$$

Integrating this over space gives the result

$$H=\int d^{3}x\left[-\frac{\hbar^{2}}{2m}\psi^{*}\nabla^{2}\psi+V\left(\mathbf{x},t\right)\psi^{*}\psi\right] \tag{3}$$

$$=\int d^{3}x\,\psi^{*}\left[-\frac{\hbar^{2}}{2m}\nabla^{2}\psi+V\left(\mathbf{x},t\right)\psi\right] \tag{4}$$

$$=\left\langle \psi\left|\hat{H}\right|\psi\right\rangle \tag{5}$$

We can work out the energy-momentum tensor for the Schrödinger Lagrangian.

$$T_{\mu\nu}=-g_{\mu\nu}\mathcal{L}+\sum_{\sigma}\frac{\partial\mathcal{L}}{\partial\left(\psi_{\sigma,\mu}\right)}\frac{\partial\psi_{\sigma}}{\partial x^{\nu}} \tag{6}$$

In this case, there are two independent fields: ψ and ψ^{*}, so the tensor comes out to (raising the second index):

$$T_{\mu\nu}=-g_{\mu\nu}\mathcal{L}+\frac{\partial\mathcal{L}}{\partial\left(\partial\psi/\partial x_{\mu}\right)}\frac{\partial\psi}{\partial x^{\nu}}+\frac{\partial\mathcal{L}}{\partial\left(\partial\psi^{*}/\partial x_{\mu}\right)}\frac{\partial\psi^{*}}{\partial x^{\nu}} \tag{7}$$

The form in Greiner's eqn (10) raises the ν index. It also interchanges μ and ν on the RHS, so he states

$$T_{\mu}^{\nu}=-\delta_{\mu}^{\nu}\mathcal{L}+\frac{\partial\mathcal{L}}{\partial\left(\partial\psi/\partial x^{\nu}\right)}\frac{\partial\psi}{\partial x^{\mu}}+\frac{\partial\mathcal{L}}{\partial\left(\partial\psi^{*}/\partial x^{\nu}\right)}\frac{\partial\psi^{*}}{\partial x^{\mu}} \tag{8}$$

Although it's not obvious from the definition 6, it seems that in most cases the energy-momentum tensor is indeed symmetric (see, for example, Wikipedia). For the Schrödinger Lagrangian 1, we can show by direct calculation that $T_{\mu\nu}$ is symmetric for spatial components. From 6

$$T_{ij}=-g_{ij}\mathcal{L}+\frac{\partial\mathcal{L}}{\partial\left(\partial\psi/\partial x_{i}\right)}\frac{\partial\psi}{\partial x^{j}}+\frac{\partial\mathcal{L}}{\partial\left(\partial\psi^{*}/\partial x_{i}\right)}\frac{\partial\psi^{*}}{\partial x^{j}} \tag{9}$$

$$=-g_{ij}\mathcal{L}-\frac{\hbar^{2}}{2m}\left[-\frac{\partial\psi^{*}}{\partial x^{i}}\frac{\partial\psi}{\partial x^{j}}-\frac{\partial\psi}{\partial x^{i}}\frac{\partial\psi^{*}}{\partial x^{j}}\right] \tag{10}$$

$$=T_{ji} \tag{11}$$

The minus signs inside the brackets come from $\frac{\partial\psi}{\partial x_i}=-\frac{\partial\psi}{\partial x^i}$ for spatial coordinates.

However, it is not symmetric for mixed time and space coordinates. For example, using 6

$$T_{01}=\frac{\partial\mathcal{L}}{\partial\left(\partial\psi/\partial x_{0}\right)}\frac{\partial\psi}{\partial x^{1}}+\frac{\partial\mathcal{L}}{\partial\left(\partial\psi^{*}/\partial x_{0}\right)}\frac{\partial\psi^{*}}{\partial x^{1}} \tag{12}$$

$$=\frac{\partial\mathcal{L}}{\partial\dot{\psi}}\frac{\partial\psi}{\partial x^{1}}+\frac{\partial\mathcal{L}}{\partial\dot{\psi}^{*}}\frac{\partial\psi^{*}}{\partial x^{1}} \tag{13}$$

$$=-\frac{\hbar}{2i}\left(\psi^{*}\frac{\partial\psi}{\partial x^{1}}-\psi\frac{\partial\psi^{*}}{\partial x^{1}}\right) \tag{14}$$

$$T_{10}=\frac{\partial\mathcal{L}}{\partial\left(\partial\psi/\partial x_{1}\right)}\frac{\partial\psi}{\partial x^{0}}+\frac{\partial\mathcal{L}}{\partial\left(\partial\psi^{*}/\partial x_{1}\right)}\frac{\partial\psi^{*}}{\partial x^{0}} \tag{15}$$

$$=-\frac{\partial\mathcal{L}}{\partial\left(\partial\psi/\partial x\right)}\dot{\psi}+\frac{\partial\mathcal{L}}{\partial\left(\partial\psi^{*}/\partial x\right)}\dot{\psi}^{*} \tag{16}$$

$$=-\frac{\hbar^{2}}{2m}\left(-\frac{\partial\psi^{*}}{\partial x^{1}}\dot{\psi}-\frac{\partial\psi}{\partial x^{1}}\dot{\psi}^{*}\right) \tag{17}$$

Greiner then shows (using his version of T_{μ}^{ν} in 8) in his eqn (11) that

$$T_{0}^{0}=\frac{\hbar^{2}}{2m}\nabla\psi^{*}\cdot\nabla\psi+V\left(\mathbf{x},t\right)\psi^{*}\psi=\mathcal{H} \tag{18}$$

Thus T_{0}^{0} is the energy density.
Greiner also calculates the energy flux, which he says is defined by

$$\mathbf{S} = e_1 T_0^{\ 1} + e_2 T_0^{\ 2} + e_3 T_0^{\ 3} \tag{19}$$

$$= -\frac{\hbar^2}{2m_0}\left(\dot{\psi}^* \nabla\psi + \dot{\psi}\nabla\psi^*\right) \tag{20}$$

Also, the momentum density

$$\mathbf{p} = e_1 T_1^{\ 0} + e_2 T_2^{\ 0} + e_3 T_3^{\ 0} \tag{21}$$

$$= \frac{i\hbar}{2}\left(\psi^* \nabla\psi - \psi\nabla\psi^*\right) \tag{22}$$

This is $-m\mathbf{J}$, where **J** is the 3-d probability current.

Using the original definition of $T_{\mu\nu}$ in 6, the definitions of **S** and **p** would be swapped. I'm not sure whether the equations in Greiner are typos or whether there's something deeper going on here.