

## KLEIN-GORDON EQUATION IN SCHRÖDINGER FORM

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Reference: W. Greiner: *Relativistic Quantum Mechanics (Wave Equations)*; 3rd Edition, Springer-Verlag (2000); Section 1.6.

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Greiner shows how the Klein-Gordon equation (a DE that is second order in both position and time) can be converted to two coupled DEs that are second order in position but only first order in time. The mathematics is worked out clearly in Greiner's eqns 1.62 to 1.82 with most steps filled in, so we'll just summarize the argument here, and fill in a few gaps.

The idea is that we take the Klein-Gordon equation

$$\left( \frac{\partial^2}{c^2 \partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + \frac{m_0^2 c^2}{\hbar^2} \right) \psi = 0 \quad (1)$$

and split the wave function  $\psi$  into two parts, according to the formulas

$$\psi = \phi + \chi \quad (2)$$

$$i\hbar \frac{\partial \psi}{\partial t} = m_0 c^2 (\phi - \chi) \quad (3)$$

Greiner shows (in the unnumbered equations before eqn 1.64) that if these two functions  $\phi$  and  $\chi$  satisfy the two coupled PDEs

$$i\hbar \frac{\partial \phi}{\partial t} = -\frac{\hbar^2}{2m_0} \nabla^2 (\phi + \chi) + m_0 c^2 \phi \quad (4)$$

$$i\hbar \frac{\partial \chi}{\partial t} = \frac{\hbar^2}{2m_0} \nabla^2 (\phi + \chi) - m_0 c^2 \chi \quad (5)$$

then these two equations can be combined to give the Klein-Gordon equation 1 again.

The two coupled PDEs can be combined into a single matrix equation if we use the matrices

$$\tau_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \tau_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \quad \tau_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (6)$$

These matrices are mathematically the same as the Pauli matrices, and thus obey the same properties

$$\tau_i \tau_j = -\tau_j \tau_i = i\tau_k \quad i, j, k = 1, 2, 3 \text{ \& cyclic perms.} \quad (7)$$

$$\tau_i^2 = \mathbf{1} \quad (8)$$

We define the vector

$$\Psi \equiv \begin{bmatrix} \phi \\ \chi \end{bmatrix} \quad (9)$$

and the matrix Hamiltonian (valid for free particles)

$$H_f \equiv (\tau_3 + i\tau_2) \frac{\mathbf{p}^2}{2m_0} + \tau_3 m_0 c^2 \quad (10)$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \frac{\mathbf{p}^2}{2m_0} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} m_0 c^2 \quad (11)$$

Then the matrix equation is

$$\left( i\hbar \frac{\partial}{\partial t} - H_f \right) \Psi = 0 \quad (12)$$

Thus the Klein-Gordon equation is represented in the same form as the Schrödinger equation.

Using the matrix products

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (13)$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^2 = \mathbf{1} \quad (14)$$

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (15)$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad (16)$$

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (17)$$

we find, on squaring  $H_f$ :

$$H_f^2 = c^2 \mathbf{p}^2 + m_0^2 c^4 \quad (18)$$

The  $m_0 c^2$  factor is erroneously shown as a subscript to  $\tau_3$  on the LHS of Greiner's eqn 1.68.

which is the usual relativistic formula for the energy of a particle. In Greiner's eqn 1.70 it is shown that each component of the vector  $\Psi$  in 9 satisfies the Klein-Gordon equation on its own.

Earlier, we've seen that the density  $\rho'$  and current  $\mathbf{j}'$  given by

$$\rho' = \frac{ie\hbar}{2m_0c^2} \left( \psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right) \quad (19)$$

$$\mathbf{j}' = -\frac{ie\hbar}{2m_0} (\psi^* \nabla \psi - \psi \nabla \psi^*) \quad (20)$$

can be interpreted as the charge density and current. In eqns 1.71 and 1.72, Greiner shows that these have the forms

$$\rho' = e\Psi^\dagger \tau_3 \Psi \quad (21)$$

$$\mathbf{j}' = \frac{e\hbar}{2im_0} \left[ \Psi^\dagger \tau_3 (\tau_3 + i\tau_2) \nabla \Psi - \nabla \Psi^\dagger \tau_3 (\tau_3 + i\tau_2) \Psi \right] \quad (22)$$

For a single, charged free particle, the total charge is  $\pm e$ , so we must have

$$\int \rho' d^3x = \int e\Psi^\dagger \tau_3 \Psi d^3x = \pm e \quad (23)$$

Plugging 9 and  $\tau_3$  into 21, this gives the condition

$$\int \Psi^\dagger \tau_3 \Psi d^3x = \int (\phi\phi^* - \chi\chi^*) d^3x = \pm 1 \quad (24)$$

For a free particle, we can propose a solution

$$\Psi = \begin{bmatrix} \phi \\ \chi \end{bmatrix} = A \begin{bmatrix} \phi_0 \\ \chi_0 \end{bmatrix} \exp \left[ \frac{i}{\hbar} (\mathbf{p} \cdot \mathbf{x} - Et) \right] \quad (25)$$

where  $A$  is a normalization constant, and  $\phi_0$  and  $\chi_0$  are parameters that depend on the energy. Inserting this solution into 12 by using 11 leads to the usual relativistic energy-momentum relation

$$E = \pm c \sqrt{p^2 + m_0^2 c^2} = \pm E_p \quad (26)$$

[If you're wondering why the determinant of Greiner's eqn 1.77 must vanish, as he says, consider a general equation in two variables, such as

$$ax + by = 0 \quad (27)$$

$$cx + dy = 0 \quad (28)$$

If we want a non-trivial solution (that is,  $x \neq 0$  and  $y \neq 0$ ), then from the first equation  $x = -by/a$ . Substituting into the second, we get

$$\left(-\frac{bc}{a} + d\right)y = 0 \quad (29)$$

Thus if  $y \neq 0$ , then  $bc = ad$ , which makes the determinant  $ad - bc = 0$ .] For  $E = +E_p$ , we can write the solution as

$$\Psi^{(+)} = A_{(+)} \begin{bmatrix} \phi_0^{(+)} \\ \chi_0^{(+)} \end{bmatrix} \exp \left[ \frac{i}{\hbar} (\mathbf{p} \cdot \mathbf{x} - E_p t) \right] \quad (30)$$

Greiner shows in the eqns following 1.78 that we can choose

$$\chi_0^{(+)} = m_0 c^2 - E_p \quad (31)$$

This results in

$$\phi_0^{(+)} = m_0 c^2 + E_p \quad (32)$$

$$A_{(+)} = \frac{1}{\sqrt{4m_0 c^2} \sqrt{L^3 E_p}} \quad (33)$$

Recall that we're treating the system in a finite volume which is a cube of side length  $L$ .

We can do the same calculations for  $E = -E_p$ , where we get the solution

$$\Psi^{(-)} = A_{(-)} \begin{bmatrix} \phi_0^{(-)} \\ \chi_0^{(-)} \end{bmatrix} \exp \left[ \frac{i}{\hbar} (\mathbf{p} \cdot \mathbf{x} + E_p t) \right] \quad (34)$$

with the results

$$\chi_0^{(-)} = m_0 c^2 + E_p \quad (35)$$

$$\phi_0^{(-)} = m_0 c^2 - E_p \quad (36)$$

$$A_{(-)} = \frac{1}{\sqrt{4m_0 c^2} \sqrt{L^3 E_p}} \quad (37)$$

In the nonrelativistic limit, Greiner shows in eqns 1.81 and 1.82 that the solutions reduce to (for  $\frac{v}{c} \ll 1$ ):

$$\Psi^{(+)} \rightarrow \frac{1}{\sqrt{L^3}} \begin{bmatrix} 1 \\ -\frac{1}{4} \frac{v^2}{c^2} \end{bmatrix} \quad (38)$$

$$\Psi^{(-)} \rightarrow \frac{1}{\sqrt{L^3}} \begin{bmatrix} -\frac{1}{4} \frac{v^2}{c^2} \\ 1 \end{bmatrix} \quad (39)$$

The state  $\Psi^{(+)}$  represents a particle with positive charge, and we can see that in the nonrelativistic limit, this state is almost entirely represented

by the  $\phi$  component. Likewise, the state  $\Psi^{(-)}$  represents a particle with negative charge and is represented by the  $\chi$  component.

#### PINGBACKS

Pingback: Klein-Gordon equation in the Feshbach-Villars representation

Pingback: Klein-Gordon equation in Schrödinger form - Lagrangian, energy-momentum

Pingback: Klein-Gordon equation in Feshbach-Villars form - Operator transformations