KLEIN-GORDON EQUATION IN SCHRÖDINGER FORM - LAGRANGIAN, ENERGY-MOMENTUM TENSOR

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Reference: W. Greiner: *Relativistic Quantum Mechanics (Wave Equations)*; 3rd Edition, Springer-Verlag (2000); Section 1.8; Exercise 1.7.

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Greiner's Exercise 1.7 introduces the Lagrangian density and energy-momentum tensor for the Schrödinger representation of the Klein-Gordon equation. (In the text, he says that it's for the Feshbach-Villars representation, but all the equations refer to the two-component wave function Ψ rather than the function Φ which is used in the Feshbach-Villars form.)

As usual, the Lagrangian density is just stated (one of these days I'd like to find out if it is possible to actually *derive* a Lagrangian rather than conjuring them out of thin air) to be

$$\mathcal{L}=i\hbar\bar{\Psi}\partial_{t}\Psi-\frac{\hbar^{2}}{2m_{0}}\nabla\bar{\Psi}\left(\tau_{3}+i\tau_{2}\right)\nabla\Psi-m_{0}c^{2}\bar{\Psi}\tau_{3}\Psi \tag{1}$$

where the τ_i matrices are essentially the Pauli matrices

$$\tau_{1}=\left[\begin{array}{cc}0 & 1\\ 1 & 0\end{array}\right];\quad\tau_{2}=\left[\begin{array}{cc}0 & -i\\ i & 0\end{array}\right];\quad\tau_{3}=\left[\begin{array}{cc}1 & 0\\ 0 & -1\end{array}\right]\quad\mathbf{1}=\left[\begin{array}{cc}1 & 0\\ 0 & 1\end{array}\right] \tag{2}$$

and

$$\bar{\Psi}\equiv\Psi^{\dagger}\tau_{3} \tag{3}$$

By using the usual technique of varying the action integral, Greiner shows that the Euler-Lagrange equations yield the Schrödinger representation of the Klein-Gordon equation. That is, variation of $\bar{\Psi}$ yields the equation

$$\partial_{\nu}\frac{\partial\mathcal{L}}{\partial\left(\partial_{\nu}\bar{\Psi}_{\alpha}\right)}-\frac{\partial\mathcal{L}}{\partial\bar{\Psi}_{\alpha}}=0 \tag{4}$$

which gives

$$i\hbar\partial_{t}\Psi_{\alpha}=-\frac{\hbar^{2}}{2m_{0}}\left(\tau_{3}+i\tau_{2}\right)\nabla^{2}\Psi_{\alpha}+m_{0}c^{2}\tau_{3}\Psi_{\alpha} \tag{5}$$

where $\alpha = 1,2$ refers to one of the two components of Ψ. Variation of Ψ gives the conjugate equation

$$-i\hbar\partial_t\bar{\Psi}_\alpha = -\frac{\hbar^2}{2m_0}(\tau_3 + i\tau_2)\nabla^2\bar{\Psi}_\alpha + m_0c^2\tau_3\bar{\Psi}_\alpha \tag{6}$$

These two equations are both forms of the Schrödinger representation of the Klein-Gordon equation, which is

Note that the 2×2 matrices τ_3 and $i\tau_2$ are both real, so their complex conjugates give the same matrix.

$$\left(i\hbar\frac{\partial}{\partial t} - H_f\right)\Psi = 0 \tag{7}$$

with

$$H_f \equiv (\tau_3 + i\tau_2)\frac{\hbar^2\nabla^2}{2m_0} + \tau_3 m_0c^2 \tag{8}$$

We've seen that an integral of the form

$$\langle L\rangle = \int \Psi^\dagger\tau_3 L\Psi d^3x \tag{9}$$

is real if the operator L satisfies the generalized hermiticity condition

$$L^H \equiv \tau_3 L^\dagger \tau_3 = L \tag{10}$$

In eqn 1.111, Greiner shows that H_f satisfies this condition. The operator $i\hbar\partial_t$ has the same effect as H_f so it is generally hermitian as well. As Greiner shows in Ex. 1.7, the general hermiticity of H_f ensures that the action integral is real.

The energy-momentum tensor is given by

$$T_{\mu\nu} = -g_{\mu\nu}\mathcal{L} + \sum_\sigma \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_\sigma)}\frac{\partial\psi_\sigma}{\partial x^\nu} \tag{11}$$

where the index σ ranges over the independent fields. In this case, the fields are Ψ and $\bar{\Psi}$, so we have

$$T_{\mu\nu} = -g_{\mu\nu}\mathcal{L} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi)}\partial_\nu\Psi + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\bar{\Psi}\right)}\partial_\nu\bar{\Psi} \tag{12}$$

The component T_{00} gives the energy density, and Greiner shows in eqns (4) and (5) that the total energy is given by

$$E = \int T_{00} d^3x \tag{13}$$

$$= \int \mathbf{\Psi}^\dagger \tau_3 H_f \mathbf{\Psi} d^3x \tag{14}$$

$$= \left\langle H_f \right\rangle \tag{15}$$

where the last line follows from the definition of the mean value of an operator in 9.