

## KLEIN-GORDON EQUATION IN FESHBACH-VILLARS FORM - OPERATOR TRANSFORMATIONS AND SOLUTIONS

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Reference: W. Greiner: *Relativistic Quantum Mechanics (Wave Equations)*; 3rd Edition, Springer-Verlag (2000); Section 1.8; Exercises 1.8 - 1.9.

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In the Schrödinger representation of the Klein-Gordon equation, the expectation value of an operator is given by

$$\langle L \rangle = \int \Psi^\dagger \tau_3 L \Psi d^3x \quad (1)$$

where  $\Psi$  is the two-component wave function and the  $\tau_i$  are essentially the Pauli matrices

$$\tau_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \tau_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \quad \tau_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (2)$$

In this representation, an operator  $L$  that represents an observable must satisfy the generalized hermiticity condition

$$L^H \equiv \tau_3 L^\dagger \tau_3 = L \quad (3)$$

This reduces to the ordinary hermiticity condition  $L^\dagger = L$  if  $[\tau_3, L] = 0$ .

In the Feshbach-Villars representation, the Schrödinger form is transformed by the operator  $U$ :

$$U \equiv \frac{(m_0c^2 + E_p) - \tau_1 (m_0c^2 - E_p)}{\sqrt{4m_0c^2E_p}} \mathbf{1} \quad (4)$$

This operator has the inverse

$$U^{-1} = \frac{(m_0c^2 + E_p) + \tau_1 (m_0c^2 - E_p)}{\sqrt{4m_0c^2E_p}} \mathbf{1} \quad (5)$$

Greiner shows in eqn 1.112 that the form of an operator  $L$  in the Feshbach-Villars representation is obtained from the transformation

$$L_\Phi = ULU^{-1} \quad (6)$$

In Exercise 1.8, Greiner shows in detail that the F-B form of the Schrödinger Hamiltonian  $H_f$  is obtained by applying this transformation to

$$H_f \equiv (\tau_3 + i\tau_2) \frac{\mathbf{p}^2}{2m_0} + \tau_3 m_0 c^2 \quad (7)$$

After a lot of calculation, and the use of the properties of the  $\tau_i$  matrices, he shows that  $H_\Phi$  reduces to

$$H_\Phi = U H_f U^{-1} = \tau_3 E_p \quad (8)$$

where  $E_p$  is the energy of the eigenstate with momentum  $\mathbf{p}$ , and is just a number, not an operator. Written out in matrix form this is

$$H_\Phi = \begin{bmatrix} E_p & 0 \\ 0 & -E_p \end{bmatrix} \quad (9)$$

The Klein-Gordon equation for a free particle in the F-B representation therefore takes the simple form

$$i\hbar \frac{\partial \Phi}{\partial t} = \tau_3 E_p \Phi \quad (10)$$

The solutions of this equation are shown in Greiner's Exercise 1.9, beginning with

$$i\hbar \frac{\partial}{\partial t} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} E_p \phi_1 \\ -E_p \phi_2 \end{bmatrix} \quad (11)$$

Since the components of  $\Phi$  are eigenstates of the momentum operator, we have

$$\phi_i = e^{i\mathbf{p}\cdot\mathbf{r}/\hbar} \vartheta_i \quad (12)$$

where the  $\vartheta_i$ s contain the time dependence. They must therefore satisfy

$$i\hbar \frac{\partial}{\partial t} \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \end{bmatrix} = \begin{bmatrix} E_p \vartheta_1 \\ -E_p \vartheta_2 \end{bmatrix} \quad (13)$$

The solutions are just the usual exponentials

$$\vartheta_1 = N_1 e^{-iE_p t/\hbar} \quad (14)$$

$$\vartheta_2 = N_2 e^{+iE_p t/\hbar} \quad (15)$$

To find the  $N_i$ s, recall that the overall wave function is normalized according to

$$\int \Psi^\dagger \tau_3 \Psi d^3x = \int \Phi^\dagger \tau_3 \Phi d^3x = \pm 1 \quad (16)$$

The inclusion of the  $\tau_3$  matrix means that this integral becomes

$$\int \Phi^\dagger \tau_3 \Phi d^3x = \int (\phi_1^\dagger \phi_1 - \phi_2^\dagger \phi_2) d^3x \quad (17)$$

$$= \int (|N_1|^2 - |N_2|^2) d^3x \quad (18)$$

$$= L^3 (|N_1|^2 - |N_2|^2) \quad (19)$$

$$= V (|N_1|^2 - |N_2|^2) \quad (20)$$

$$= \pm 1 \quad (21)$$

Remember that we're enclosing the system in a cube of side length  $L$ , so these integrals are finite.

where  $V = L^3$  is the volume of the cube. Therefore the normalization condition is

$$(|N_1|^2 - |N_2|^2) = \pm \frac{1}{\sqrt{V}} \quad (22)$$

In the F-B representation, the top component  $\phi_1 = \Phi^{(+)}$  of the wave function represents a particle with positive charge and the bottom component  $\phi_2 = \Phi^{(-)}$ , a particle with negative charge. The full solutions are

$$\Phi^{(+)} = \frac{1}{\sqrt{V}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{i\mathbf{p}\cdot\mathbf{r}/\hbar} e^{-iE_p t/\hbar} \quad (23)$$

$$\Phi^{(-)} = \frac{1}{\sqrt{V}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{i\mathbf{p}\cdot\mathbf{r}/\hbar} e^{+iE_p t/\hbar} \quad (24)$$

At the end of this exercise, Greiner states that we can have a linear combination of these two solutions, such as

$$n_1 \Phi^{(+)} + n_2 \Phi^{(-)} \quad (25)$$

with

$$|n_1|^2 - |n_2|^2 = \pm 1 \quad (26)$$

and that this linear combination represents a particle with charge  $\pm 1$  depending on the value of  $|n_1|^2 - |n_2|^2$ . I'm not terribly clear on the meaning of this, since I thought the whole point of the F-B representation was to separate the two charges into separate components. In any case, he goes on, in eqns 1.114 to 1.118, to say that we can then specify the eigenstate with momentum  $\mathbf{p}$  and charge  $\lambda = \pm 1$  as  $\Phi_{\mathbf{p},\lambda}$  where

$$\mathbf{p} = \left\{ \frac{2\pi\hbar}{L} n_i \right\} \quad (27)$$

where  $n_i = 0, \pm 1, \pm 2, \dots$ , with the integer  $n_i$  determined by the boundary conditions on the wave inside the volume  $V$ . A general solution for a many-particle system is then a combination of these solutions, so that

$$\Phi = \sum_{\mathbf{p}, \lambda} a_{\mathbf{p}, \lambda} \Phi_{\mathbf{p}, \lambda} \quad (28)$$

$$= \sum_{\mathbf{p}} (a_{\mathbf{p}, +1} \Phi_{\mathbf{p}, +1} + a_{\mathbf{p}, -1} \Phi_{\mathbf{p}, -1}) \quad (29)$$

This is a sum over all momentum and charge states, with coefficients  $a_{\mathbf{p}, \lambda}$  that are determined from the usual orthonormality condition:

$$\int \Phi_{\mathbf{p}, \lambda}^\dagger \tau_3 \Phi_{\mathbf{p}', \lambda'} d^3x = \lambda \delta_{\mathbf{p}\mathbf{p}'} \delta_{\lambda\lambda'} \quad (30)$$

To find  $a_{\mathbf{p}, \lambda}$ , relabel the summation index  $\mathbf{p}$  to  $\mathbf{p}'$  in 29, then multiply both sides by  $\Phi_{\mathbf{p}, \lambda}$  and integrate. This gives the condition

$$a_{\mathbf{p}, \lambda} = \int \Phi_{\mathbf{p}, \lambda}^\dagger \tau_3 \Phi d^3x \quad (31)$$

The total charge is then, using 30

$$Q = e \int \Phi^\dagger \tau_3 \Phi d^3x \quad (32)$$

$$= e \sum_{\mathbf{p}} (|a_{\mathbf{p}, +1}|^2 - |a_{\mathbf{p}, -1}|^2) \quad (33)$$

$$= \pm Ne \quad (34)$$

where  $N$  is the net number of elementary charges  $e$  in the system.

I'm not entirely clear on how we can guarantee that 33 will always give an integer times  $e$ . Presumably in the overall wave function 29, the choices for  $\Phi$  are restricted in some way so that the system always contains an integral number of charges. This would seem to require that each  $a_{\mathbf{p}, \lambda}$  is the square root of an integer.

Greiner sometimes uses a lowercase  $\phi$  which is a typo - all  $\Phi$ s should be uppercase.