

KLEIN-GORDON EQUATION - INTERACTION WITH ELECTROMAGNETIC FIELD

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Reference: W. Greiner: *Relativistic Quantum Mechanics (Wave Equations)*; 3rd Edition, Springer-Verlag (2000); Section 1.9.

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The Klein-Gordon equation is a relativistic wave equation valid for spinless particles. For a free particle, the equation is

$$(p^\mu p_\mu - m_0 c^2) \psi = 0 \quad (1)$$

where p^μ is the four-momentum operator. We've seen that the density ρ' and current \mathbf{j}' can be interpreted as representing charge and current densities, so the Klein-Gordon equation can represent a spinless particle with a positive, negative, or zero charge.

Since we can represent a charged particle, the natural next question to ask is how we can introduce an electromagnetic field with which the particle can interact. Classically and nonrelativistically, the Hamiltonian for a particle with charge e in an electromagnetic field is given by

$$H = \frac{|\mathbf{p} - e\mathbf{A}/c|^2}{2m} + e\phi \quad (2)$$

where \mathbf{A} is the magnetic vector potential and ϕ is the electric potential. We can generalize this by introducing the four-potential A^μ where

$$A^0 = \phi \quad (3)$$

$$A^\mu = A_i; \quad i = x, y, z \quad (4)$$

The Klein-Gordon equation which includes interaction with an electromagnetic field is obtained by replacing

$$p^\mu \rightarrow p^\mu - \frac{e}{c} A^\mu \quad (5)$$

in 1. This gives us

$$\left(p^\mu - \frac{e}{c} A^\mu\right) \left(p_\mu - \frac{e}{c} A_\mu\right) \psi = m_0 c^2 \psi \quad (6)$$

To calculate the charge and current densities, we multiply 6 on the left by ψ^* and the complex conjugate of 6 by ψ and subtract the latter from the

Keep in mind that 1 is a relativistic equation and 2 is nonrelativistic, so the energy has different forms in the two equations.

former. Greiner does this in detail leading to his eqn 1.123 which identifies the four-current as

$$j_\nu = \frac{i\hbar e}{2m_0} (\psi^* \partial_\nu \psi - \psi \partial_\nu \psi^*) - \frac{e^2}{m_0 c} A_\nu \psi \psi^* = \{c\rho', -\mathbf{j}'\} \quad (7)$$

The spatial components are $-\mathbf{j}'$ rather than \mathbf{j}' because we're using the covariant (lower) index ν rather than the contravariant (upper) index. Explicitly, this gives the charge and current densities as

$$\rho' = j_0 = \frac{i\hbar e}{2m_0 c^2} \left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right) - \frac{e^2}{m_0 c^2} A_0 \psi \psi^* \quad (8)$$

$$\mathbf{j}' = -\frac{i\hbar e}{2m_0} (\psi^* \nabla \psi - \psi \nabla \psi^*) - \frac{e^2}{m_0 c} \mathbf{A} \psi \psi^* \quad (9)$$

We can compare these forms with those for a free particle (no EM field):

$$\rho' = \frac{i\hbar e}{2m_0 c^2} \left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right) \quad (10)$$

$$\mathbf{j}' = -\frac{i\hbar e}{2m_0} (\psi^* \nabla \psi - \psi \nabla \psi^*) \quad (11)$$

The two forms are the same except that when we introduce an EM field, both the charge and current density pick up a term that depends on the potential. Greiner discusses this following eqn 1.126, although it appears that a proper understanding requires field theory. The charge density 8 can actually be of the opposite sign to the charge e of the particle being considered, and this is ascribed to particle-antiparticle pair production in strong fields.

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The extra factor of c in the denominator of ρ' comes from $\partial_0 = \partial/c\partial t$.