KLEIN-GORDON EQUATION - INTERACTION WITH ELECTROMAGNETIC FIELD

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: W. Greiner: *Relativistic Quantum Mechanics (Wave Equations)*; 3rd Edition, Springer-Verlag (2000); Section 1.9.

Post date: 13 Dec 2017.

The Klein-Gordon equation is a relativistic wave equation valid for spinless particles. For a free particle, the equation is

$$\left(p^{\mu}p_{\mu}-m_{0}c^{2}\right)\psi=0 \tag{1}$$

where p^{μ} is the four-momentum operator. We've seen that the density ρ' and current $\mathbf{j}'$ can be interpreted as representing charge and current densities, so the Klein-Gordon equation can represent a spinless particle with a positive, negative, or zero charge.

Since we can represent a charged particle, the natural next question to ask is how we can introduce an electromagnetic field with which the particle can interact. Classically and nonrelativistically, the Hamiltonian for a particle with charge e in an electromagnetic field is given by

$$H=\frac{\left|\mathbf{p}-e\mathbf{A}/c\right|^{2}}{2m}+e\phi \tag{2}$$

where $\mathbf{A}$ is the magnetic vector potential and ϕ is the electric potential. We can generalize this by introducing the four-potential A^{μ} where

$$A^{0}=\phi \tag{3}$$

$$A^{\mu}=A_{i};\ i=x,y,z \tag{4}$$

The Klein-Gordon equation which includes interaction with an electromagnetic field is obtained by replacing

$$p^{\mu}\rightarrow p^{\mu}-\frac{e}{c}A^{\mu} \tag{5}$$

in 1. This gives us

$$\left(p^{\mu}-\frac{e}{c}A^{\mu}\right)\left(p_{\mu}-\frac{e}{c}A_{\mu}\right)\psi=m_{0}c^{2}\psi \tag{6}$$

Keep in mind that 1 is a relativistic equation and 2 is nonrelativistic, so the energy has different forms in the two equations.

To calculate the charge and current densities, we multiply 6 on the left by ψ^{*} and the complex conjugate of 6 by ψ and subtract the latter from the former. Greiner does this in detail leading to his eqn 1.123 which identifies the four-current as

$$j_\nu = \frac{i\hbar e}{2m_0}\left(\psi^*\partial_\nu\psi - \psi\partial_\nu\psi^*\right) - \frac{e^2}{m_0 c}A_\nu\psi\psi^* = \left\{c\rho', -\mathbf{j}'\right\} \tag{7}$$

The spatial components are $-\mathbf{j}'$ rather than $\mathbf{j}'$ because we're using the covariant (lower) index ν rather than the contravariant (upper) index. Explicitly, this gives the charge and current densities as

$$\rho' = j_0 = \frac{i\hbar e}{2m_0 c^2}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right) - \frac{e^2}{m_0 c^2}A_0\psi\psi^* \tag{8}$$

$$\mathbf{j}' = -\frac{i\hbar e}{2m_0}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) - \frac{e^2}{m_0 c}\mathbf{A}\psi\psi^* \tag{9}$$

We can compare these forms with those for a free particle (no EM field):

The extra factor of c in the denominator of ρ' comes from $\partial_0 = \partial/c\partial t$.

$$\rho' = \frac{i\hbar e}{2m_0 c^2}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right) \tag{10}$$

$$\mathbf{j}' = -\frac{i\hbar e}{2m_0}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) \tag{11}$$

The two forms are the same except that when we introduce an EM field, both the charge and current density pick up a term that depends on the potential. Greiner discusses this following eqn 1.126, although it appears that a proper understanding requires field theory. The charge density 8 can actually be of the opposite sign to the charge e of the particle being considered, and this is ascribed to particle-antiparticle pair production in strong fields.

PINGBACKS

Pingback: Klein-Gordon equation with Coulomb potential

Pingback: Klein-Gordon equation - invariance under electromagnetic gauge transformations

Pingback: Klein-Gordon equation with scalar 1/r potential