KLEIN-GORDON EQUATION WITH COULOMB POTENTIAL

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Reference: W. Greiner: *Relativistic Quantum Mechanics (Wave Equations)*; 3rd Edition, Springer-Verlag (2000); Section 1.9, Exercises 1.10, 1.11.

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Now that we have the Klein-Gordon equation with an eletromagnetic field, we can try to solve the equation when the field consists of the Coulomb potential for a point charge. The K-G equation is

$$\left(p^{\mu}-\frac{e}{c}A^{\mu}\right)\left(p_{\mu}-\frac{e}{c}A_{\mu}\right)\psi=m_0c^2\psi \tag{1}$$

In this case, the four-potential is

$$eA^{\mu}=\{V(r),0,0,0\} \tag{2}$$

That is, the magnetic vector potential is zero, and the Coulomb potential is given by

$$A^0(r)=-\frac{Ze}{r} \tag{3}$$

where e is the (positive) elementary charge and Z is the number of particles, each of which has charge e. The solution of the K-G equation is similar to that for the hydrogen atom when using the Schrödinger equation. However, note that we cannot use the K-G equation to build a relativistic model of hydrogen since the K-G equation applies only to spinless particles, and the electron in the hydrogen atom has spin-$\frac{1}{2}$. We will need to use the Dirac equation when studying the electron.

As we've already gone through the calculations for the hydrogen atom and Greiner spells out most of the steps in the solution of the K-G equation in his exercises 1.10 and 1.11, I don't want to go through all the details again here. Rather, I'll try to summarize the main points by comparing them with the solution of the Schrödinger equation for hydrogen.

The main K-G equation to be solved is

$$\left[(\varepsilon-V(r))^2-m_0^2c^4+\hbar^2c^2\nabla^2\right]\psi=0 \tag{4}$$

where ε is the energy. Since $V(r)$ depends only on the radial distance, we can use the usual separation of variables technique to get a solution of form

$$\psi(\mathbf{r}) = u(r)Y(\theta,\phi) \tag{5}$$

where Y is a spherical harmonic and u is the radial function. After solving the angular equation, we find that the radial equation is

Greiner calls the main radial function u while in the hydrogen atom, this function is called R.

$$\left[-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{l(l+1)}{r^2}\right]u(r) = \frac{(\varepsilon-V)^2-m_0^2c^4}{\hbar^2c^2}u(r) \tag{6}$$

At this stage, the corresponding equation for hydrogen is

$$\left[-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{l(l+1)}{r^2}\right]R(r) = -\frac{2m}{\hbar^2}(V(r)-E)R(r) \tag{7}$$

Thus the LHS of the two equations is the same; it is only the RHS that differs. Greiner now introduces the function

$$u(r) \equiv \frac{R(r)}{r} \tag{8}$$

This results in the ODE

At this point in the hydrogen atom, the function $u(r) \equiv rR(r)$ is introduced. Thus the definitions of u and R are reversed in the two solutions.

$$\left[\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}+k^2\right]R(r) = 0 \tag{9}$$

where

$$k^2 \equiv \frac{(\varepsilon-V)^2-m_0^2c^4}{\hbar^2c^2} \tag{10}$$

If we now use

$$V(r) = eA^0 = -\frac{Ze^2}{r} \tag{11}$$

and insert this into 10, we have

$$k^2 = \frac{\varepsilon^2-m_0^2c^4}{\hbar^2c^2}+\frac{Z^2e^4}{\hbar^2c^2r^2}+\frac{2Ze^2\varepsilon}{\hbar^2c^2r} \tag{12}$$

Introducing the fine structure constant

$$\alpha \equiv \frac{e^2}{\hbar c} \tag{13}$$

we find that 9 becomes

$$\left[\frac{d^2}{dr^2}-\frac{l(l+1)-Z^2\alpha^2}{r^2}+\frac{2Z\alpha\varepsilon}{\hbar c r}-\frac{m_0^2c^4-(\varepsilon-V)^2}{\hbar^2c^2}\right]R_l(r) = 0 \tag{14}$$

From this point onwards, the solution is quite similar to that for the hydrogen atom and you can see Greiner's exercise 1.11 for the details. The result is that the stationary states are given by

$$u(\rho) = N\rho^{\mu-1/2}e^{-\rho/2}f(\rho) \tag{15}$$

where

$$\rho \equiv 2\frac{\sqrt{m_0^2c^4-\varepsilon^2}}{\hbar c}r \tag{16}$$

$$\mu \equiv \sqrt{\left(l+\frac{1}{2}\right)^2 - Z^2\alpha^2} \tag{17}$$

The function $f(\rho)$ is, as in the case of hydrogen, found as a series in ρ:

$$f(\rho) = \sum_{n=0}^{\infty} a_n\rho^n \tag{18}$$

where the coefficients are determined by the recursion relation

$$a_m = \frac{a+m-1}{m(d+m-1)}a_{m-1} \tag{19}$$

with the constants d and a defined as

$$a \equiv \mu + \frac{1}{2} - \frac{2Z\alpha\varepsilon}{\hbar c\beta} \tag{20}$$

$$d \equiv 2\mu + 1 \tag{21}$$

Notice that the energy ε enters the recursion relation in the parameter a. As with hydrogen, we find that f as defined in 18 diverges for large ρ (and hence large r) unless the series terminates, and it is this termination condition which gives us the energy levels, which turn out to be

Greiner uses the symbol c for what I've called d, but this can be confused with the speed of light, so I've used d instead.

$$\varepsilon_{nl} = -m_0c^2\left[1+\frac{Z^2\alpha^2}{\left(n+\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2-Z^2\alpha^2}\right)^2}\right]^{-1/2} \tag{22}$$

This analysis applies to a spin-0 particle, such as a pion, in a Coulomb potential.

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