KLEIN-GORDON EQUATION - INVARIANCE UNDER ELECTROMAGNETIC GAUGE TRANSFORMATIONS

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Reference: W. Greiner: *Relativistic Quantum Mechanics (Wave Equations)*; 3rd Edition, Springer-Verlag (2000); Section 1.10.

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The Klein-Gordon equation for a particle in an electromagnetic field is given by

$$\left(p^{\mu}-\frac{e}{c}A^{\mu}\right)\left(p_{\mu}-\frac{e}{c}A_{\mu}\right)\psi=m_{0}c^{2}\psi \tag{1}$$

where A^{μ} is the electromagnetic four-potential and $p^{\mu}=i\hbar\frac{\partial}{\partial x_{\mu}}$ is the four-momentum. The classical equations of electromagnetism (Maxwell's equations) are invariant if we impose a gauge transformation on the potential. In four-vector notation, the most general gauge transformation is given by

$$A'_{\mu}(x)=A_{\mu}(x)+\frac{\partial\chi(x)}{\partial x^{\mu}} \tag{2}$$

where $\chi(x)$ is an arbitrary scalar function of the four-vector $x^{\mu}=(ct,\mathbf{x})$.

What happens if we insert this gauge transformation into the Klein-Gordon equation 1? We get

$$\left(p^{\mu}-\frac{e}{c}A^{\mu}-\frac{e}{c}\frac{\partial\chi(x)}{\partial x_{\mu}}\right)\left(p_{\mu}-\frac{e}{c}A_{\mu}-\frac{e}{c}\frac{\partial\chi(x)}{\partial x^{\mu}}\right)\psi=m_{0}c^{2}\psi \tag{3}$$

or, using Greiner's form:

$$g^{\mu\nu}\left(p_{\nu}-\frac{e}{c}A_{\nu}-\frac{e}{c}\frac{\partial\chi(x)}{\partial x^{\nu}}\right)\left(p_{\mu}-\frac{e}{c}A_{\mu}-\frac{e}{c}\frac{\partial\chi(x)}{\partial x^{\mu}}\right)\psi=m_{0}c^{2}\psi \tag{4}$$

We can write this transformed equation in the same form as 1 if we define the transformed wave function

$$\psi'\equiv\psi e^{ie\chi/\hbar c} \tag{5}$$

We can see this by replacing ψ by ψ' in the expression

$$\left(p_\mu-\frac{e}{c}A_\mu\right)\psi \to \left(p_\mu-\frac{e}{c}A_\mu\right)\psi' \tag{6}$$

$$=\left(i\hbar\frac{\partial}{\partial x^\mu}-\frac{e}{c}A_\mu\right)\left(\psi e^{ie\chi/\hbar c}\right) \tag{7}$$

$$=e^{ie\chi/\hbar c}\left(i\hbar\frac{\partial}{\partial x^\mu}-\frac{e}{c}A_\mu\right)\psi+i\hbar\frac{ie}{\hbar c}\frac{\partial\chi}{\partial x^\mu}\psi e^{ie\chi/\hbar c} \tag{8}$$

$$=e^{ie\chi/\hbar c}\left(p_\mu-\frac{e}{c}A_\mu-\frac{e}{c}\frac{\partial\chi(x)}{\partial x^\mu}\right)\psi \tag{9}$$

In the second line, the operator in the first pair of parentheses operates on the product $\psi e^{ie\chi/\hbar c}$ in the second pair of parentheses, so we use the product rule to work out the derivative and get the next line. In other words

$$\left(p_\mu-\frac{e}{c}A'_\mu\right)\psi=\left(p_\mu-\frac{e}{c}A_\mu\right)\psi'=\left(p_\mu-\frac{e}{c}A_\mu\right)\left(\psi e^{ie\chi/\hbar c}\right) \tag{10}$$

The important point of this transformation is that it effectively multiplies the wave function by a phase factor $e^{ie\chi/\hbar c}$ which is independent of the wave function ψ itself. That is, once we specify the gauge transformation function $\chi(x)$, *all* functions transform the same way, by picking up this phase factor. We can therefore extend the argument as follows.

First, we define a bit of shorthand for the operators:

$$B_\mu\equiv\left(p_\mu-\frac{e}{c}A_\mu\right) \tag{11}$$

$$B'_\mu\equiv\left(p_\mu-\frac{e}{c}A'_\mu\right) \tag{12}$$

Then from 10

$$e^{ie\chi/\hbar c}B'_\mu(\psi)=B_\mu\left(\psi e^{ie\chi/\hbar c}\right) \tag{13}$$

The parentheses here indicate what the operators B'_μ and B_μ operate on. Now the LHS of 3 can be rewritten as

$$B'^\mu\left(B'_\mu(\psi)\right) \tag{14}$$

Using 13 we have

$$e^{ie\chi/\hbar c}B'^\mu\left(B'_\mu(\psi)\right)=B^\mu\left(B'_\mu(\psi)\,e^{ie\chi/\hbar c}\right) \tag{15}$$

However, also from 13:

$$B'_\mu(\psi)=e^{-ie\chi/\hbar c}B_\mu\left(\psi e^{ie\chi/\hbar c}\right) \tag{16}$$

so we get

$$e^{ie\chi/\hbar c}B'^{\mu}\left(B'_{\mu}\left(\psi\right)\right) = B^{\mu}\left(e^{-ie\chi/\hbar c}B_{\mu}\left(\psi e^{ie\chi/\hbar c}\right)e^{ie\chi/\hbar c}\right) \tag{17}$$

$$= B^{\mu}\left(B_{\mu}\left(\psi e^{ie\chi/\hbar c}\right)\right) \tag{18}$$

$$= g^{\mu\nu}B_{\nu}\left(B_{\mu}\left(\psi e^{ie\chi/\hbar c}\right)\right) \tag{19}$$

In other words, replacing B and B' by their definitions, we have that the transformed K-G equation 3 can be written as

It's very important to keep track of what each operator operates on, which is shown by the parentheses in these equations.

$$\left(p^{\mu} - \frac{e}{c}A^{\mu}\right)\left(p_{\mu} - \frac{e}{c}A_{\mu}\right)\psi' = m_0 c^2 \psi' \tag{20}$$

Thus applying a gauge transformation to the K-G equation just gives the same equation but with a wave function ψ' that is the original wave function ψ multiplied by a phase factor that depends only on $\chi(x)$, and is the same for all wave functions. Since this phase factor cancels out in any calculation of an observable quantity, it doesn't affect the physics and we can say that the K-G equation is invariant under gauge transformations.

We can extend the argument to higher powers the same way, but in summary, the result is that (using Greiner's shorthand notation in his eqn 1.136):

$$e^{ie\chi/\hbar c}\left(p_{\mu} - \frac{e}{c}A'_{\mu}\right)^{n}\psi = \left(p_{\mu} - \frac{e}{c}A_{\mu}\right)^{n}\psi' = \left(p_{\mu} - \frac{e}{c}A_{\mu}\right)^{n}\left(\psi e^{ie\chi/\hbar c}\right) \tag{21}$$

For any function $f\left(p_{\mu} - \frac{e}{c}A_{\mu}\right)$ that can be expanded in a power series, this result means that applying a gauge transformation to the function $f\left(p_{\mu} - \frac{e}{c}A_{\mu}\right)\psi$ and get

$$e^{ie\chi/\hbar c}f\left(p_{\mu} - \frac{e}{c}A'_{\mu}\right)\left(\psi\right) = f\left(p_{\mu} - \frac{e}{c}A_{\mu}\right)\left(\psi e^{ie\chi/\hbar c}\right) \tag{22}$$