KLEIN-GORDON EQUATION WITH COULOMB POTENTIAL - CHARGED SPHERE

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Reference: W. Greiner: *Relativistic Quantum Mechanics (Wave Equations)*; 3rd Edition, Springer-Verlag (2000); Section 1.11, Example 1.13.

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We've looked at the Klein-Gordon equation with a Coulomb potential due to a point charge and applied this to a pion around a nucleus with charge Ze. A more realistic model of the nucleus is as a uniformly charged sphere of radius a. From classical electrostatics, the potential energy inside such a sphere is given by (in CGS units):

$$V(r)=\varepsilon A_0(r)=-\frac{Ze^2}{2a}\left(3-\frac{r^2}{a^2}\right) \tag{1}$$

We can plug this potential into the K-G equation

$$\left[\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}+k^2\right]R(r)=0 \tag{2}$$

where

$$k^2\equiv\frac{(\varepsilon-V)^2-m_0^2c^4}{\hbar^2c^2} \tag{3}$$

which gives Greiner's equation (2) in Example 1.13. To get this equation, Greiner has replaced ε by the energy E and the elementary charge e by the fine structure constant

$$\alpha=\frac{e^2}{\hbar c} \tag{4}$$

As a result

$$k^2 = \frac{1}{\hbar^2 c^2}\left(E + \frac{Z\alpha\hbar c}{2a}\left(3 - \frac{r^2}{a^2}\right)\right)^2 - \frac{m_0^2 c^2}{\hbar^2} \tag{5}$$

$$= \left(\frac{E}{\hbar c} + \frac{Z\alpha}{2a}\left(3 - \frac{r^2}{a^2}\right)\right)^2 - \frac{m_0^2 c^2}{\hbar^2} \tag{6}$$

$$= \left[\left(\frac{E}{\hbar c} + \frac{3Z\alpha}{2a}\right) - \frac{Z\alpha}{2a}\frac{r^2}{a^2}\right]^2 - \frac{m_0^2 c^2}{\hbar^2} \tag{7}$$

$$= \left(\frac{E}{\hbar c} + \frac{3Z\alpha}{2a}\right)^2 - \frac{m_0^2 c^2}{\hbar^2} - 2\left(\frac{E}{\hbar c} + \frac{3Z\alpha}{2a}\right)\frac{Z\alpha}{2a^3}r^2 + \left(\frac{Z\alpha}{2a^3}\right)^2 r^4 \tag{8}$$

Using the symbols

$$A \equiv \frac{E}{\hbar c} + \frac{3Z\alpha}{2a} \tag{9}$$

$$B \equiv A^2 - \frac{m_0^2 c^2}{\hbar^2} \tag{10}$$

$$C \equiv \frac{Z\alpha}{2a^3} \tag{11}$$

2 becomes

$$\left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + B - 2ACr^2 + C^2 r^4\right] R(r) = 0 \tag{12}$$

As usual, we try a series solution

$$R = r^{l+1}\sum_{n'=0}^{\infty} b_{n'} r^{2n'} \tag{13}$$

If we insert this into 12 we get Greiner's eqn (5). This leads to the recursion relation given by Greiner's eqn (7):

We use an exponent of $2n'$ since only even powers of r occur in 12.
Note that the last term in Greiner's (5) should be multiplied by C^2 and not just C.

$$b_{n'} = \frac{Bb_{n'-1} - 2ACb_{n'-2} + C^2 b_{n'-3}}{4n'l + (2n'+1)\,2n'} \tag{14}$$

For large n', we see that the coefficients $b_{n'}$ get smaller and smaller, tending to 0 as $1/(n')^2$. As the series $\sum \frac{1}{n^2}$ converges, we would expect the series 13 to converge for all finite r (I realize this isn't a rigorous derivation, but hopefully you get the idea.) Since this solution is valid only for $r \le a$, we can't impose the condition that the series must terminate at some finite value of n'. Rather, the quantization condition arises from the requirement that the wave function and its derivative must both be continuous at the boundary $r = a$.

Outside the sphere, the potential is the same as that for a point charge of magnitude Ze, so the solution for $r>a$ is given by our earlier result

$$R(\rho)=N'W_{\lambda,\mu}(\rho) \tag{15}$$

where W is a Whittaker function (see the earlier post for details) with

$$\beta \equiv 2\frac{\sqrt{m_0^2c^4-\varepsilon^2}}{\hbar c} \tag{16}$$

$$\rho \equiv \beta r \tag{17}$$

$$\mu \equiv \sqrt{\left(l+\frac{1}{2}\right)^2-Z^2\alpha^2} \tag{18}$$

$$\lambda \equiv \frac{2Z\alpha\varepsilon}{\hbar c\beta} \tag{19}$$

We therefore need to solve the condition given by Greiner's eqn (8):

$$\frac{W'_{\lambda,\mu}(\beta a)}{W_{\lambda,\mu}(\beta a)}=\frac{\sum_{n'}(2n'+l+1)\,b_{n'}a^{2n'+l}}{\sum_{n'}b_{n'}a^{2n'+l+1}} \tag{20}$$

The numerators on each side represent the derivative of the wave function for $r<a$ (LHS) and $r>a$ (RHS), while the denominators represent the corresponding wave functions themselves. Remember that in the earlier solution, W is given as a function of $\rho=\beta a$ and the derivative $W'(\beta a)$ is the derivative with respect to r, evaluated at $r=a$. Presumably we would need to solve this numerically and artificially truncate the two series on the RHS in order to get an answer. Greiner gives a few examples of solutions for a pionic atom.

Greiner points out that, for the point nucleus that we considered earlier, if $l=0$ then, in order to keep μ real, we must have

$$Z^2\alpha^2<\frac{1}{4} \tag{21}$$

which limits Z to values less than 68. He claims that in the sphere of charge nucleus considered here, we can find solutions for larger values of Z when $l=0$. However, since μ still appears in the condition 20, it's not entirely clear that an imaginary value of μ gives acceptable results. Presumably it does, but it would seem quite complicated to show this.

Greiner's example graphs are actually for a sphere of charge that obeys the Fermi charge distribution

$$\rho(r) = \frac{N}{1 + \exp\left[(4\ln 3)(r-c)/t\right]} \tag{22}$$

where c and t are constants governing the shape of the curve (they are *not* the speed of light and time!). This density gives a reasonably constant value up to around $r = c - \frac{t}{2}$, at which point the density falls to zero over a distance t.