

KLEIN-GORDON EQUATION WITH COULOMB POTENTIAL - CHARGED SPHERE

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Reference: W. Greiner: *Relativistic Quantum Mechanics (Wave Equations)*; 3rd Edition, Springer-Verlag (2000); Section 1.11, Example 1.13.

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We've looked at the Klein-Gordon equation with a Coulomb potential due to a point charge and applied this to a pion around a nucleus with charge Ze . A more realistic model of the nucleus is as a uniformly charged sphere of radius a . From classical electrostatics, the potential energy inside such a sphere is given by (in CGS units):

$$V(r) = \epsilon A_0(r) = -\frac{Ze^2}{2a} \left(3 - \frac{r^2}{a^2} \right) \quad (1)$$

We can plug this potential into the K-G equation

$$\left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + k^2 \right] R(r) = 0 \quad (2)$$

where

$$k^2 \equiv \frac{(\epsilon - V)^2 - m_0^2 c^4}{\hbar^2 c^2} \quad (3)$$

which gives Greiner's equation (2) in Example 1.13. To get this equation, Greiner has replaced ϵ by the energy E and the elementary charge e by the fine structure constant

$$\alpha = \frac{e^2}{\hbar c} \quad (4)$$

As a result

$$k^2 = \frac{1}{\hbar^2 c^2} \left(E + \frac{Z\alpha \hbar c}{2a} \left(3 - \frac{r^2}{a^2} \right) \right)^2 - \frac{m_0^2 c^2}{\hbar^2} \quad (5)$$

$$= \left(\frac{E}{\hbar c} + \frac{Z\alpha}{2a} \left(3 - \frac{r^2}{a^2} \right) \right)^2 - \frac{m_0^2 c^2}{\hbar^2} \quad (6)$$

$$= \left[\left(\frac{E}{\hbar c} + \frac{3Z\alpha}{2a} \right) - \frac{Z\alpha}{2a} \frac{r^2}{a^2} \right]^2 - \frac{m_0^2 c^2}{\hbar^2} \quad (7)$$

$$= \left(\frac{E}{\hbar c} + \frac{3Z\alpha}{2a} \right)^2 - \frac{m_0^2 c^2}{\hbar^2} - 2 \left(\frac{E}{\hbar c} + \frac{3Z\alpha}{2a} \right) \frac{Z\alpha}{2a^3} r^2 + \left(\frac{Z\alpha}{2a^3} \right)^2 r^4 \quad (8)$$

Using the symbols

$$A \equiv \frac{E}{\hbar c} + \frac{3Z\alpha}{2a} \quad (9)$$

$$B \equiv A^2 - \frac{m_0^2 c^2}{\hbar^2} \quad (10)$$

$$C \equiv \frac{Z\alpha}{2a^3} \quad (11)$$

2 becomes

$$\left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + B - 2ACr^2 + C^2 r^4 \right] R(r) = 0 \quad (12)$$

As usual, we try a series solution

$$R = r^{l+1} \sum_{n'=0}^{\infty} b_{n'} r^{2n'} \quad (13)$$

If we insert this into 12 we get Greiner's eqn (5). This leads to the recursion relation given by Greiner's eqn (7):

$$b_{n'} = \frac{Bb_{n'-1} - 2ACb_{n'-2} + C^2 b_{n'-3}}{4n'l + (2n'+1)2n'} \quad (14)$$

For large n' , we see that the coefficients $b_{n'}$ get smaller and smaller, tending to 0 as $1/(n')^2$. As the series $\sum \frac{1}{n^2}$ converges, we would expect the series 13 to converge for all finite r (I realize this isn't a rigorous derivation, but hopefully you get the idea.) Since this solution is valid only for $r \leq a$, we can't impose the condition that the series must terminate at some finite value of n' . Rather, the quantization condition arises from the requirement that the wave function and its derivative must both be continuous at the boundary $r = a$.

We use an exponent of $2n'$ since only even powers of r occur in 12. Note that the last term in Greiner's (5) should be multiplied by C^2 and not just C .

Outside the sphere, the potential is the same as that for a point charge of magnitude Ze , so the solution for $r > a$ is given by our earlier result

$$R(\rho) = N'W_{\lambda,\mu}(\rho) \quad (15)$$

where W is a Whittaker function (see the earlier post for details) with

$$\beta \equiv 2 \frac{\sqrt{m_0^2 c^4 - \epsilon^2}}{\hbar c} \quad (16)$$

$$\rho \equiv \beta r \quad (17)$$

$$\mu \equiv \sqrt{\left(l + \frac{1}{2}\right)^2 - Z^2 \alpha^2} \quad (18)$$

$$\lambda \equiv \frac{2Z\alpha\epsilon}{\hbar c\beta} \quad (19)$$

We therefore need to solve the condition given by Greiner's eqn (8):

$$\frac{W'_{\lambda,\mu}(\beta a)}{W_{\lambda,\mu}(\beta a)} = \frac{\sum_{n'} (2n' + l + 1) b_{n'} a^{2n'+l}}{\sum_{n'} b_{n'} a^{2n'+l+1}} \quad (20)$$

The numerators on each side represent the derivative of the wave function for $r < a$ (LHS) and $r > a$ (RHS), while the denominators represent the corresponding wave functions themselves. Remember that in the earlier solution, W is given as a function of $\rho = \beta a$ and the derivative $W'(\beta a)$ is the derivative with respect to r , evaluated at $r = a$. Presumably we would need to solve this numerically and artificially truncate the two series on the RHS in order to get an answer. Greiner gives a few examples of solutions for a pionic atom.

Greiner points out that, for the point nucleus that we considered earlier, if $l = 0$ then, in order to keep μ real, we must have

$$Z^2 \alpha^2 < \frac{1}{4} \quad (21)$$

which limits Z to values less than 68. He claims that in the sphere of charge nucleus considered here, we can find solutions for larger values of Z when $l = 0$. However, since μ still appears in the condition 20, it's not entirely clear that an imaginary value of μ gives acceptable results. Presumably it does, but it would seem quite complicated to show this.

Greiner's example graphs are actually for a sphere of charge that obeys the Fermi charge distribution

$$\rho(r) = \frac{N}{1 + \exp[(4 \ln 3)(r - c)/t]} \quad (22)$$

where c and t are constants governing the shape of the curve (they are *not* the speed of light and time!). This density gives a reasonably constant value up to around $r = c - \frac{t}{2}$, at which point the density falls to zero over a distance t .